Characterizing the Distortion of Some Simple Euclidean Embeddings

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Abstract

We consider two related families of problems. First we consider the embedding of finite point sets on a circle into one or more lines, or finite point sets on a sphere onto one or more planes. Next we consider the problem of embedding $N+1$ points from $\mathbb{R}^{K+1}$ into $\mathbb{R}^K$ where all but one of the $N+1$ points are in $\mathbb{R}^K$. Given such point sets, in the worst case, how much distortion must necessarily be incurred, by the best embedding?

1 Introduction

Various authors have studied the problem of minimizing the distortion of embedding points from one metric space into another metric space. In this work we consider two related families of problems. First we consider the embedding of finite point sets on a circle into one or more lines, or finite point sets on a sphere onto one or more planes. Next we consider the problem of embedding $N+1$ points from $\mathbb{R}^{K+1}$ into $\mathbb{R}^K$ where all but one of the $N+1$ points are in $\mathbb{R}^K$. Given such point sets, in the worst case, how much distortion must necessarily be incurred, by the best embedding? In the case of the $N+1$ points, how does the maximum distortion compare to the case where an unbounded number of points can lie outside any particular hyperplane of $\mathbb{R}^{K+1}$? Questions of this nature are important in many application areas, from data compression to machine learning.

Notation: Let Π be an embedding of one metric space, $\mathcal{M}_1$ into a second metric space, $\mathcal{M}_2$. Let $d_1(x,y)$ denote the distance between two points $x,y \in \mathcal{M}_1$ and let $d_2(x,y)$ denote the distance between two points $x,y \in \mathcal{M}_2$.

Definition: Let P be a finite point set in a metric space $\mathcal{M}_1$, and let $\Pi : P \to \mathcal{M}_2$ be a mapping (embedding) of P into $\mathcal{M}_2$. Then the **distortion** of the mapping Π, $\mathrm{Dist}(\Pi)$ is given by

$$\mathrm{Dist}(\Pi) = \max\left(\max_{x,y\in P} \frac{d_2(\Pi(x),\Pi(y))}{d_1(x,y)}, \max_{x,y\in P} \frac{d_1(x,y)}{d_2(\Pi(x),\Pi(y))}\right).$$

2 Background and Related Work

A fundamental reference that discusses the Lipschitz extension theorem of Kirszbraun (see next section) and the now classical Johnson-Lindenstrauss-Schechtman Lemmas is [4]. [1] and [2] study embedding metric spaces into a line, and into the two-dimensional plane. Our work is most closely related to [3], which discusses online metric embeddings. [5] and [6] are two older works that study the embedding of finite metric spaces into low dimensional Euclidean spaces.

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3 Embeddings Points on a Circle into a Line and Points on a Sphere into a Plane

Definition: Call a set of N points on the unit sphere S^K **dense** if the radius of the largest empty cap is of size $O\left(\frac{1}{N^{1/K}}\right)$.

Unless otherwise stated, all metrics are assumed to be the Euclidean metric of the ambient spaces. Badiou et al. [2] showed that any embedding of N points on the sphere into a plane has distortion $O(\sqrt{N})$ and that a dense set of points on the unit sphere embeds into $\mathbb{R}^2$ with distortion $\Theta(\sqrt{N})$. The proof of the latter uses the Borsuk-Ulam Theorem together with Kirszbraun's Theorem [4], which says that a Liptschitz embedding of a subset of a Hilbert Space into another Hilbert Space can be extended to a Liptschitz embedding of the full space, with the same Liptschitz constant. The same arguments can be used to show that any embedding of N points on a circle into a line has distortion $O(N)$ and that a dense set of points on the unit circle embeds into the line with distortion that is $\Theta(N)$.

4 Embeddings Points on a Circle into Multiple Lines and Points on a Sphere into Multiple Planes

We first consider the problem of embedding points on a circle into two lines.

Lemma 1 *A set of N points on a circle can be embedded into two lines selected by the problem solver with distortion that is $O(\sqrt{N})$.*

Proof. Consider an origin-centered disk of unit radius and a set P of N points on the disk. To this set of points add their antipodal points $-P$. Among the points in $P \cup -P$ there is some pair of adjacent points p, p' that are $\Omega(\frac{1}{N})$ from one another. The antipodals to these points are also adjacent to one another with the same separation. Split the circle with a diametric cut that passes between p and p', and also between $-p$ and $-p'$.

Half of the points of $P \cup -P$ will be on one side of this cut and half on the other side. Without loss of generality suppose the cut is the line $y = 0$. Now consider the two lines $y = \frac{1}{\sqrt{N}}$ and $y = -\frac{1}{\sqrt{N}}$. Define an embedding Π of the points on the circle to the two lines as follows. Embed the points on the top half of the circle, say that in left to right order they are $p, ..., -p'$, onto $y = \frac{1}{\sqrt{N}}$, also in left to right order, so that their sequential distances from one another are the same as their geodetic distances on the circle. Then embed the points on the bottom half of the circle, i.e. $-p, ..., p'$, this time in their natural right to left ordering, onto $y = -\frac{1}{\sqrt{N}}$, so that their sequential distances from one another are again the same as their geodetic distances on the circle, and moreover, such that the image of $-p$ is directly below the image of $-p'$.

For points q, q' that are mapped to the same line, Π introduces just a constant amount of distortion since the geodetic distance along the circle is an $O(1)$-approximation to the Euclidean distance. To see this formally we need show that the ratio of the length of an arc of the unit circle to the associated chord length is bounded by a constant. On a unit circle the length of an arc associated with a central angle θ is also θ, while the associated chord length is $\sqrt{2 - 2\cos\theta} = 2\sin\frac{\theta}{2}$. So we must determine the supremum of

$$f(\theta) = \frac{\theta}{2\sin\frac{\theta}{2}}. \tag{1}$$

By L'Hôpital's Rule,

$$\lim_{\theta \to 0} f(\theta) = 1, \tag{2}$$

and

$$f'(\theta) = \frac{2\sin\frac{\theta}{2} - \theta\cos\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} \tag{3}$$

and the numerator is never 0. It follows that the supremum of f is the maximum of 1 (the effective value of $f(0)$) and $f(\pi) = \frac{\pi}{2}$, establishing that the geodetic distance along the circle is an $O(1)$-approximation to the Euclidean distance.

Thus the biggest distortion is either the distortion introduced at $\Pi(p), \Pi(p')$ (equivalently, at $\Pi(-p), \Pi(-p')$), which is potentially the most pronounced expansion, or by the most pronounced contraction, which can be no more substantial than if there were points at $(0,1), (0,-1)$ with $(0,1)$ embedding into the line $y = \frac{1}{\sqrt{N}}$ and $(0,-1)$ embedding into the line $y = -\frac{1}{\sqrt{N}}$.

However the distortion at $\Pi(p), \Pi(p')$ is

$$O\left(\frac{\frac{1}{\sqrt{N}}}{\frac{1}{N}}\right) = O(\sqrt{N}), \tag{4}$$

while the distortion assuming there were points at $(0,1), (0,-1)$ would be

$$O\left(\frac{1}{\frac{1}{\sqrt{N}}}\right) = O(\sqrt{N}). \tag{5}$$

Thus the lemma is established. □

Lemma 2 *A set of N points on a circle can be embedded into three lines selected by the problem solver with constant distortion.*

Proof. We consider the unit circle, C, and an associated circumscribed equilateral triangle, T. We map the N points on C to N geodetically proportionally spaced points on T, respecting the ordering of the points on C. Call this map Π. We show that Π can neither expand nor contract distances by too much. The proof that Π does not expand too much breaks down into a series of three fairly trivial observations, namely: (i) The geodesic distance on the circle is an $O(1)$-APX to the associated Euclidean distance, (ii) the geodesic distance on the triangle is an $O(1)$-APX to the geodesic distance on the circle, and (iii) the Euclidean distance between two points on a triangle is never greater than the geodesic distance on the triangle (obvious). That Π is at most a constant factor expansion means that the Euclidean distance on the triangle is at most a constant factor expansion to the associated Euclidean distance on the circle. The result follows by the transitivity of the $O(1)$-APX relation if we can establish (i) and (ii).

We established the truth of (i) in our proof of Lemma 1. (ii) is even easier since the approximation ratio is just the ratio of the associated perimeters, which is $\frac{3\sqrt{3}}{\pi}$. The fact that Π expands by at most a constant factor follows.

To show that Π contracts by at most a constant factor, it suffices that (a) the geodesic distance on the circle does not decrease distances relative to the Euclidean distance on the circle, (b) the geodesic distance on the triangle does not decrease distances relative to the geodesic distance on the circle, and (c) the Euclidean distance on the triangle does not contract distances by more than a constant factor relative to the geodesic distance on the triangle. (a) and (b) are obvious. For (c) consider Figure 1. By the law of cosines,

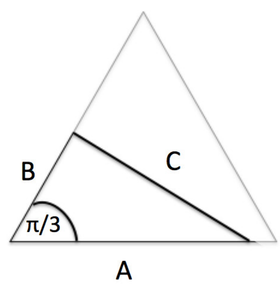

Figure 1. Comparison of the Euclidean distance, C, between two points on an equilateral triangle, and the geodesic distance $A + B$.

$$C^2 = A^2 + B^2 - 2AB\cos\frac{\pi}{3} = A^2 + B^2 - AB. \tag{6}$$

Now, without loss of generality, assume that $A \geq B$ and that $A = m + \varepsilon$, $B = m - \varepsilon$ (where $m = \frac{A+B}{2}$ and $\varepsilon = \frac{A-B}{2}$). Then

$$\begin{aligned} C^2 &= (m+\varepsilon)^2 + (m-\varepsilon)^2 + (m+\varepsilon)(m-\varepsilon) \\ &= m^2 + 3\varepsilon^2, \end{aligned}$$

and so $C \geq \frac{A+B}{2}$. Thus the Euclidean distance contracts by no more than a factor of 2 relative to the geodesic distance on the equilateral triangle and so Π contracts by at most a constant factor, and the lemma is established. $\square$

We observe that it is not possible to extend Lemma 2 to an annulus of constant thickness. If it were possible to make such an extension then it would be possible to embed points on an ε-thick annulus into a disk with constant distortion. However, in what is a variant of a rather usual argument/counterexample, consider N points in the annulus contained within a $\sqrt{N}$ x $\sqrt{N}$ square grid, each grid point at a distance of $\delta = \varepsilon/N$ from the next. If we embed these points onto the circle so that the distortion of each point with its neighbor on the circle gets constant distortion, then they must be placed at distance no smaller than $k\delta$ from one another from some constant k. However, the two furthest apart points on the circle will be approximately $Nk\delta$ from one another, while they started at distance no greater than $\sqrt{2N}\delta$ from one another. Thus they incur a distortion of at least $\sqrt{N}$.

It is conceivable, however, that Lemma 1 can be extended to cover the case of an annulus of constant thickness.

Definition: Say that a set of N points on the sphere is distributed **approximately uniformly** if the geodetic distance between any two points in the set is $\Omega(\frac{1}{\sqrt{N}})$ and there is no empty patch (cap) of radius $\Omega(\frac{1}{\sqrt{N}})$.

We believe the next lemma holds for an arbitrary set of N points on the sphere but the best we can prove at present is the following:

Lemma 3 *An approximately uniformly distributed set of N points on a sphere can be embedded into two planes selected by the problem solver with distortion that is $O(N^{1/4})$.*

Proof. Place N points approximately uniformly on a unit radius sphere. We will embed the points on the surface of the sphere onto two planes at $z = \pm\frac{1}{N^{1/4}}$. Points on the bottom half of he sphere will be embedded onto the $z = -\frac{1}{N^{1/4}}$ plane via the following two step process: (1) Project each such point p first to the $z = -1$ plane via the unique line through p that makes a 45° angle with the z-axis. Map the south pole to the south pole. (2) Map points from the $z = -1$ plane to the $z = -\frac{1}{N^{1/4}}$ via vertical projection. Do analogously to embed points on the northern hemisphere into the $z = \frac{1}{N^{1/4}}$ plane.

As in the case of the circle, projection of points on a common hemisphere onto a plane incurs a constant amount of distortion. For points near the equator and near the south pole there is essentially no distortion while the distortion is maximized for points midway between the equator and the south pole, when the distortion is easily seen to be $\sqrt{2}$.

Thus the biggest possible distortion either arises at the mapping of potential points (p_N, p_S) where p_N denotes a point at the north (top) pole of the sphere and p_S denotes its antipodal point, or at pairs of points $(p_{(\theta,\phi)_{\text{top}}}, p_{(\theta',\phi')_{\text{bottom}}})$, which denote a pair of points initially as close as possible but on opposite hemispheres, and which therefore get embedded into different planes. The distortion in the case of (p_N, p_S) is:

$$\text{Dist} = \frac{2}{\frac{2}{N^{1/4}}} = N^{1/4}. \tag{7}$$

While the distortion at $(p_{(\theta,\phi)_{\text{top}}}, p_{(\theta',\phi')_{\text{bottom}}})$ is

$$\text{Dist} = \frac{\frac{2}{N^{1/4}}}{O(\frac{1}{\sqrt{N}})} = O(N^{1/4}), \tag{8}$$

establishing the lemma. $\square$

Lemma 4 *Any set of N points on the surface of a sphere can be embedded into four planes selected by the problem solver with constant distortion.*

The proof of this lemma proceeds by projecting the N points on the sphere outward onto the regular tetrahedron that has the given sphere as its inscribed sphere. One then verifies that the Euclidean distance between two projected points is both bounded above and below by a constant factor times the Euclidean distance determined by the original points. The proof, the details of which we omit, is similar in spirit, though a bit more cumbersome than the proof of Lemma 2.

5 Embedding N Points on a Line and One Point off the Line onto a Line or N Points on a Plane and One Point off the Plane onto a Plane

Lemma 5 *Consider a collection of an odd number, N, of points on a line, each point one unit from the next, together with one additional point at height $\sqrt{N}$ above the center point of the points on the line. Then any embedding of these points into a line has distortion $\Omega(\sqrt{N})$*

Proof (sketch). Label the points consecutively along the line by $P = \{p_1, ..., p_N\}$, and refer to the point above the line at distance $\sqrt{N}$ by q. Further, denote the central point among the points in P, $p_{\frac{N+1}{2}}$, by p_{cent}.

We prove the lemma by contradiction. Suppose we have a non-contracting embedding Π of $P \cup \{q\}$ into a line, which has distortion $o(\sqrt{N})$. Consider first that some of the points in P are mapped under Π to one side of $\Pi(q)$ and some to the other. It must then be the case that some pair of adjacent points p_i and p_{i+1} are mapped by Π to opposite sides of $\Pi(q)$. But if p_i is mapped to one side of $\Pi(q)$ and p_{i+1} is mapped to the other, then, by non-contraction, the distance between $\Pi(q)$ and each of its closest neighbors is at least $\sqrt{N}$ and thus $d(\Pi(p_i), \Pi(p_{i+1})) \geq 2\sqrt{N}$ so the distortion in Π is at least $2\sqrt{N}$. Thus, for Π to have distortion $o(\sqrt{N})$ all points $\Pi(p_i)$ must be to one side of $\Pi(q)$.

Since all points $\Pi(p_i)$ are to one side of $\Pi(q)$ there is a closest neighbor $\Pi(p^*)$ to $\Pi(q)$. Divide the points of P, as evenly as possible into four sequential quarters. p^* either comes from one of the outer quarters of $\{p_1,\ldots,p_N\}$ or from one of the two inner quarters. Label these quarters $P_{1/4}, P_{2/4}, P_{3/4}$ and $P_{4/4}$, respectively.

On the one hand, if $p^* \in P_{1/4} \cup P_{4/4}$, a calculation shows that the distortion in the mapping of q, p_{cent} is at least $\frac{\sqrt{N}}{2}$.

On the other hand, if $p^* \in P_{2/4} \cup P_{3/4}$, an analogous computation shows that there must be a pair of consecutive points p_i, p_{i+1} whose distortion is at least $N/2$, establishing the lemma.

Lemma 6 *Consider a set of N points inside a disk of radius $\sqrt{N}$ with largest empty subdisk of size $O(1)$, together with one additional point at height $N^{1/4}$ above the center point of the points in the disk. Then any embedding of these points into the plane has distortion $\Omega(N^{1/4})$.*

Proof (sketch). Suppose we have a non-expanding embedding Π of the N points, P, in the disk, together with the point above the center of the disk, which we again call q, into the plane. Extend Π to be a non-expanding embedding of all of $\mathbb{R}^3$ into $\mathbb{R}^2$ by Kirszbraun's Theorem. Since Π is Lipschitz (with Lipschitz constant at most 1), Π is continuous. Let p_{cent} be the centerpoint in P directly below q. Consider the image under Π of the vertical diameter of the disk $\Pi(\text{diam})$. This image is a continuous curve through $\Pi(p_{\text{cent}})$. Color the top half of $\Pi(\text{diam})$ red and the bottom half green. Now consider the image $\Pi(\text{diam})$ as the diameter turns through 180 degrees. Continue to color $\Pi(\text{top-half})$ red and $\Pi(\text{bottom-half})$ green. A straight forward argument shows that either the endpoints of the red and green halves of these curves collectively form a closed curve with $\Pi(p_{\text{cent}})$ in its interior *or* at some point in the turning of the diameter either the end point of the green curve intersects the red curve or the end point of the red curve intersects the green curve. Suppose one of these latter two cases holds, say it is that the end point of the red curve intersects the green curve. If p_{r_e} is the pre-image of the end point of the red curve at this juncture, then there is a point p_g which is the pre-image of a point along the green curve such that $d(\Pi(p_{r_e}), \Pi(p_g)) \approx 1$ while the points p_{r_e}, p_g lie along a diameter and are at least $\sqrt{N}$ apart in the pre-image. Thus, in this case, $Dist(\Pi) = \Omega(\sqrt{N})$.

On the other hand, if $\Pi(p_{\text{cent}})$ is in the interior of the image of the disk then consider $\Pi(q)$, the image of the point above p_{cent}. If $\Pi(q)$ lies inside the image of the boundary of the disk, then since Π is non-expanding there is a point of the disk that is approximately distance 1 or less from $\Pi(q)$. Since the point started at least at distance $N^{1/4}$, the incurred distortion is $\Omega(N^{1/4})$. On the other hand, if the boundary of the disk lies between $\Pi(q)$ and $\Pi(p_{\text{cent}})$ then we again find a distortion of $\Omega(N^{1/4})$.

Future Work

These results are just the first of a hoped for more detailed characterization of how one incurs distortion on a point-by-point basis embedding from one Euclidean space into another of smaller dimension. In general if N points in $\mathbb{R}^k$ can incur some maximum distortion when the points are embedded in $\mathbb{R}^{k'}$, for $k' < k$, how much distortion can be incurred from a point set of the same size N, but where all but M of the N points lie in some k'-flat, and $M = o(N)$?

Many questions also remain regarding the low distortion embedding of points on an N-sphere into hyperplanes. For the 2-sphere we have results for one, two and four planes selected by the problem solver, but how about three planes? Can one achieve lesser order of magnitude distortion using three planes than two? We currently do not know how to do this and speculate that it is not possible.

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