Strongly Monotone Drawings of Planar Graphs*

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Abstract

A straight-line drawing of a graph is called *monotone* if for each pair of vertices there exists a path which is monotonically increasing in some direction, and it is called a *strongly monotone* if the direction of monotonicity is given by the direction of the line segment connecting the two vertices.

We present algorithms to compute crossing-free strongly monotone drawings for some classes of planar graphs; namely, 3-connected planar graphs, outerplanar graphs, and 2-trees. The drawings of 3-connected planar graphs are based on primal-dual circle packings. Our drawings of outerplanar graphs depend on a new algorithm that constructs strongly monotone drawings of trees which are also convex. For trees without degree-2 vertices, these drawings are strictly convex.

1 Introduction

When reading data visualized as a drawing of a graph, a common task is to find a path between a source vertex and a target vertex. This task serves as the motivation for the following quality criterion for graph drawings.

Let Γ be a straight-line drawing of graph $G = (V, E)$. We say that a path P in Γ is *monotone with respect to* a direction (or vector) d if the orthogonal projections of the vertices of P on a line with direction d appear in the same order as in P. Drawing Γ is called *monotone* if for each pair of vertices $u, v \in V$ there is a connecting path that is monotone with respect to some direction. To support the path-finding tasks it is useful to restrict the monotone direction for each path to the direction of the line segment connecting the source and the target vertex: a path $v_1 v_2 \dots v_k$ is called *strongly monotone* if it is monotone with respect to the vector $\overrightarrow{v_1 v_k}$. Drawing Γ is called *strongly monotone* if each pair of vertices $u, v \in V$ is connected by a strongly monotone path. We are interested in strongly monotone drawings which are also planar. If crossings are allowed, then any strongly monotone drawing of a spanning tree of G yields a strongly monotone drawing of G [1]. For the results stated in this abstract we interpret monotonicity in a strict sense, i.e., we do not allow edges on the path that are orthogonal to the segment between the endpoints.

Is has been shown that every connected planar graph admits a monotone drawing on a grid of size $O(n) \times O(n^2)$ [4]. On the other hand, there exists an infinite class of 1-connected graphs that do not admit strongly monotone drawings [5]. Any tree and any 2-connected outerplanar graph has a strongly monotone drawing [5]. It is known that the area required for strongly monotone drawings of trees and binary cacti is exponential [6].

In this work, we show that any 3-connected planar graph admits a strongly monotone drawing (Section 2). Then, we answer in the affirmative the open question of Kindermann et al. [5] on whether every tree has a strongly monotone drawing which is (strictly) convex. We use this result to show that every outerplanar graph admits a strongly monotone drawing (Section 3). Finally, we prove that 2-trees can be drawn strongly monotone (Section 4). All our proofs are constructive and admit efficient drawing algorithms. Our main open question is whether every planar 2-connected graph admits a plane strongly monotone drawing. Due to space constraints we either sketch or omit the proofs; detailed proofs can be found in the full preprint version [3].

2 3-Connected Planar Graphs

In this section, we prove the following:

Theorem 1 *Every 3-connected planar graph has a strongly monotone drawing.*

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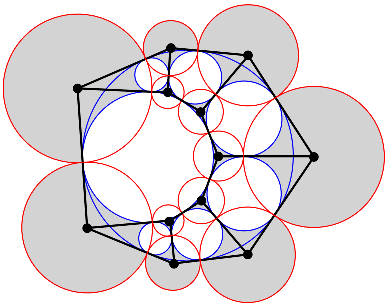

Figure 1: A primal-dual circle packing. Vertex circles in red, face circles in blue, regions of faces in white and regions of vertices in gray.

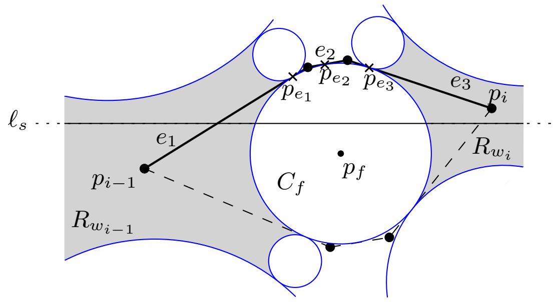

Figure 2: The path P_i connecting p_{i-1} and p_i.

We show that the straight-line drawing corresponding to a primal-dual circle packing of a graph G is strongly monotone. The theorem then follows from the fact that any 3-connected planar graph $G = (V, E)$ admits a primal-dual circle packing [2].

A *primal-dual circle packing* of a plane graph G consists of two families $\mathcal{C}_V$ and $\mathcal{C}_F$ of circles such that, there is a bijection $v \leftrightarrow C_v$ between the set V of vertices of G and circles of $\mathcal{C}_V$ and a bijection $f \leftrightarrow C_f$ between the set F of faces of G and circles of $\mathcal{C}_F$. Moreover, the following three properties hold: (**I**) The circles in the family $\mathcal{C}_V$ are interiorly disjoint and their contact graph is G, i.e., $C_u \cap C_v \neq \emptyset$ if and only if $(u, v) \in E(G)$. (**II**) If $C_o \in \mathcal{C}_F$ is the circle of the outer face o, then the circles of $\mathcal{C}_F \setminus \{C_o\}$ are interiorly disjoint while C_o contains all of them. The contact graph of $\mathcal{C}_F$ is the dual G^* of G, i.e., $C_f \cap C_g \neq \emptyset$ if and only if $(f, g) \in E(G^*)$. (**III**) The circle packings C_V and C_F are orthogonal, i.e., if $e = (u, v)$ and the dual of e is $e^* = (f, g)$, then there is a point $p_e = C_u \cap C_v = C_f \cap C_g$; moreover, the common tangents t_{e^*} of C_u, C_v and t_e of C_f, C_g cross perpendicularly in p_e.
Let a primal-dual circle packing of a graph G be given. For each vertex v, let p_v be the center of the corresponding circle C_v. By placing each vertex v at p_v, we obtain a planar straight-line drawing Γ of G. In this drawing, the edge $e = (u, v)$ is represented by the segment with end-points p_u and p_v on t_e. The face circles are inscribed circles of the faces of Γ; moreover, C_f is touching each boundary edge of the face f; see Figure 1. A straight-line drawing Γ^* of the dual G^* of G with the dual vertex of the outer face o at infinity can be obtained similarly by placing the dual vertex of each bounded face f at the center of the corresponding circle C_f. In this drawing, a dual edge $e^* = (f, o)$ is represented by the ray supported by t_{e^*} that starts at p_f and contains p_e.

We make use of a specific partition Π of the plane; Figure 1 gives an illustration. The regions of Π correspond to the vertices and the faces of G. For a vertex or face x, let D_x be the interior of the disk C_x. We define the region R_f of a bounded face f as D_f. The region R_v of a vertex v is obtained from the disk D_v by removing the intersections with the disks of bounded faces, i.e., $R_v = D_v \setminus \bigcup_{f \neq o} R_f = D_v \setminus \bigcup_{f \neq o} D_f$. To get a partition of the whole plane, we assign the complement of the already defined regions to the outer face. Note that the edge-points p_e are part of the boundary of four regions of Π. Additionally, if two regions of Π share more than one point on the boundary, then one of them is a vertex region R_v, the other is a face-region D_f, and v is incident to f in G.

We are now prepared to prove the strong monotonicity of Γ. Consider two vertices u and v and let ℓ be the line spanned by p_u and p_v. W.l.o.g., assume that ℓ is horizontal and p_u lies left of p_v. Let ℓ_s be the directed segment from p_u to p_v. Since $p_u \in R_u$ and $p_v \in R_v$, the segment ℓ_s starts and ends in these regions. In between, the segment will traverse some other regions of Π. This is true unless (u, v) is an edge of G whence the strong monotonicity for the pair is trivial. We assume *non-degeneracy* in the sense that the interior of the segment ℓ_s contains no vertex-point p_w, edge-point p_e, or face-point p_f.

Let $u = w_0, w_1, \ldots, w_k = v$ be the sequence of vertices whose region is intersected by ℓ_s, in the order of intersection from left to right and let $p_i = p_{w_i}$. We will construct a strongly monotone path P from p_u to p_v in Γ that contains $p_u = p_0, p_1, \ldots, p_k = p_v$ in this order. We show how to construct P_i, the subpath of P from p_{i-1} to p_i. Since ℓ_s may revisit a vertex-region, it is possible that $p_{i-1} = p_i$; in this case we set $P_i = p_i$. Now suppose that $p_{i-1} \neq p_i$. Non-degeneracy implies that the segment ℓ_s alternates between vertex-regions and face-regions; hence, a unique disk D_f is intersected by ℓ_s between the regions of w_{i-1} and w_i. It follows that w_{i-1} and w_i are vertices on the boundary of f. The boundary of f contains two paths from w_{i-1} to w_i. In Γ, one of these two paths from p_{i-1} to p_i is above D_f; we call it the *upper path*, the other one is below D_f, this is the *lower path*. If the center p_f of D_f lies below ℓ, we choose the upper path from p_{i-1} to p_i as P_i; otherwise, we choose the lower path.

Suppose that this rule led to the choice of the upper path; see Figure 2. The case that the lower path was chosen works analogously. We have to show that P_i is monotone with respect to ℓ_s, i.e., to the x-axis. Let $e_1, \ldots, e_r$ be the edges of this path and let $e_j = (q_{j-1}, q_j)$; in particular $q_0 = p_{i-1}$ and $q_r = p_i$. Since $R_{w_{i-1}}$ is star-shaped with center p_{i-1}, the segment connecting p_{i-1} with the first intersection point of ℓ with C_f belongs to $R_{w_{i-1}}$. Therefore, the point p_{e_1} of tangency of edge e_1 at C_f lies above ℓ. Similarly, p_{e_r} and, hence, all the points p_{e_j} lie above ℓ. Since the points $p_{e_1}, \ldots, p_{e_r}$ appear in this order on C_f and the center of C_f lies below ℓ, we obtain that their x-coordinates are increasing in this order. This sequence is interleaved with the x-coordinates of $q_0, q_1, \ldots, q_r$, whence this is also monotone. This proves that the chosen path P_i is monotone with respect to ℓ. Monotonicity also holds for the concatenation $P = P_1 + P_2 + \ldots + P_k$.

Even if degenerancy is allowed, there still exists a strongly monotone path consisting of the edges tangent to the circles intersected by l_s. This can be shown by carefully examining the arising special cases.

3 Trees and Outerplanar Graphs

Consider a straight-line, crossing-free drawing Γ of a tree and replace each edge that leads to a leaf by a ray that begins with the edge and extends accross the leaf. If all the unbounded polygonal regions in the obtained drawing Γ' are convex, then drawing Γ is called *convex*. If all angles in Γ' are less than π, then Γ is called *strictly* convex.

Kindermann et al. [5] have shown that any tree has a strongly monotone drawing and that any binary tree has a strictly convex strongly monotone drawing. They left as an open question whether every tree admits a convex strongly monotone drawing; noticing that, in the positive case, this would imply that every Halin graph has a convex strongly monotone drawing. We give an affirmative answer to this question by stating the following:

Theorem 2 *Every tree has a convex strongly monotone drawing. If the tree has no degree-2 vertex, then the drawing is strictly convex.*

The theorem can be proven by inductively generating a corresponding drawing. In the beginning some root vertex v_0 is placed in the plane and its k children are placed at the corners of a regular k-gon with center v_0. The drawing if then generated by iteratively expanding leafs while maintaining the following:

Invariant: (**I**) Every leaf is located on a corner of the convex hull of the vertices. (**II**) If $a_1, \ldots, a_\ell$ is the counterclockwise order of the leaves on the convex hull, then for $i = 1, \ldots, \ell$ the vectors $(\overrightarrow{a_i a_{i-1}})^\perp$, $\overrightarrow{p_i, a_i}$, $(\overrightarrow{a_{i+1} a_i})^\perp$ appear in counterclockwise radial order, where p_i denotes the unique vertex adjacent to a_i. (**III**) The angle between two consecutive edges incident to a vertex v is at most π and is equal to π only when v has degree two. (**IV**) Γ is strongly monotone.

We can utilize Theorem 2 to show that every outerplanar graph has a strongly monotone drawing that is *convex*, i.e. every internal face is realized as a convex region.

Theorem 3 *Every outerplanar graph has a convex strongly monotone drawing.*

Proof. Let G be an outerplanar graph with at least 2 vertices. For every vertex $v \in V$, we add two dummy vertices v', v'' and edges (v, v'), (v, v''). By construction, the resulting graph H is outerplanar and does not contain vertices of degree 2. Let Γ_H be an outerplanar drawing of H. We will construct a convex strongly monotone drawing Γ'_H of H with the same combinatorial embedding as Γ_H.

Let T be an arbitrary spanning tree of H. By construction, no vertex in T has degree 2. Thus, according to Theorem 2, T admits a strongly monotone drawing Γ_T which is strictly convex and which also preserves the order of the children for every vertex, i.e., the rotation system coincides with the one in Γ_H.

Now, we insert all the missing edges. Recall that, by removing an edge from a planar drawing, the two adjacent faces are merged. Since the drawing Γ_T of T is strictly convex and since Γ_T preserves the rotation system of Γ_H, by inserting an edge e of the graph H into Γ_T one strictly convex face is partitioned into two strictly convex faces. Furthermore, the insertion of an edge does not destroy strong monotonicity. We reinsert all edges of H iteratively. The resulting drawing Γ'_H of H is a strictly convex and strongly monotone.

Finally, we remove all the dummy vertices and obtain a strongly monotone drawing of G. Since Γ'_H has the same combinatorial embedding as Γ_H, every dummy vertex lies in the outer face. Hence, no internal face is affected by the removal of dummy vertices, and thus all interior faces remain strictly convex. ☐

4 2-Trees

A *2-tree* is a graph produced by starting with a K_3 and then repeatedly adding vertices such that each added vertex v has exactly two neighbours v_1, v_2 and there is an edge $e = (v_1, v_2)$. We say that v is *stacked* on e. In this section, we provide a proof sketch for the following theorem:

Theorem 4 *Every 2-tree admits a strongly monotone drawing.*

We begin by introducing some notation. A *drawing with bubbles* of a graph $G = (V, E)$ is a straight-line drawing of G in the plane such that, for some $E' \subseteq E$, every edge $e \in E'$ is associated with a circular region

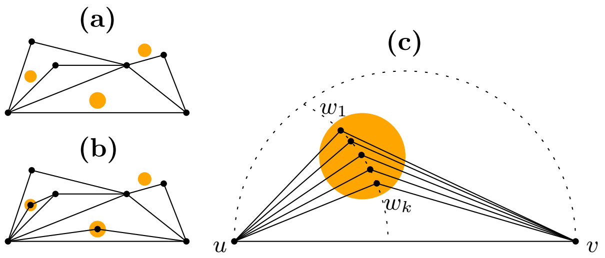

Figure 3: A drawing with bubbles (a) together with an extension (b). Stacking vertices into a bubble (c).

in the plane, called a *bubble* B_e; see Figure 3(a). An *extension* of a drawing with bubbles is a straight-line drawing that is obtained by taking some subset of edges with bubbles $E'' \subseteq E'$ and stacking one vertex on top of each edge $e \in E''$ into the corresponding bubble B_e; see Figure 3(b). (Since every bubble is associated with a unique edge we often simply say that a vertex is stacked into a bubble without mentioning the corresponding edge.) We call a drawing with bubbles Γ strongly monotone if *every* extension of Γ is strongly monotone. Note that this implies that if a vertex w is stacked on top of edge e into bubble B_e, then there exists a strongly monotone path from w to any other vertex in the drawing and, furthermore, there exists a strongly monotone path from w to any of the current bubbles, i.e., to any vertex that might be stacked into another bubble.

Every 2-tree $T = (V, E)$ can be constructed through the following iterative procedure: **1.** Start with one edge and tag it as *active*. During the entire procedure, every present edge is tagged either as active or *inactive*. **2.** Pick one active edge e and stack vertices $w_1, \ldots, w_k$ on top of this edge for some $k \geq 0$ (we note that k might equal 0). Edge e is then tagged as inactive and all new edges incident to the stacked vertices $w_1, \ldots, w_k$ are tagged as active. **3.** If there are active edges remaining, repeat Step 2.

Observe that Step 2 is performed exactly once per edge and that an according decomposition for T can always be found by the definition of 2-trees. We construct a strongly monotone drawing of T by geometrically implementing the iterative procedure described above, so that after every step of the algorithm the present part of the graph is realized as a drawing with bubbles. We maintain the following:

Invariant: After each step of the algorithm every active edge comes with a bubble and the drawing with bubbles is strongly monotone. Additionaly, for an edge $e = (u, v)$ with bubble B_e for each point $w \in B_e$, the angle $\angle(\overrightarrow{uw}, \overrightarrow{wv})$ is obtuse.

In Step 1, we arbitrarily draw the edge e_0 in the plane. Clearly, it is possible to define a bubble for e_0 that only allows obtuse angles. In Step 2, we place the vertices $w_1, \ldots, w_k$ over an edge $e = (u, v)$ as follows. The fact that stacking a vertex into B_e gives an obtuse angle allows us to place the to-be stacked vertices $w_1, \ldots w_k$ in B_e on a circular arc around u such that, for any $1 \leq i, j \leq k$, there exists a strongly monotone path between w_i and w_j; see Figure 3(c). Due to our invariant, there also exists a strongly monotone path between any of the newly stacked vertices and any vertex of an extension of the previous drawing with bubbles. Hence, after removing the bubble B_e, the resulting drawing is a strongly monotone drawing with bubbles.

In order to maintain the invariant, it remains to describe how to define the bubbles for the new active edges incident to the stacked vertices. For this purpose, we state the following Lemma 5, which enables us to define the two bubbles for the edges incident to any degree-2 vertex with an obtuse angle. The Lemma is then iteratively applied to the vertices $w_1, \ldots, w_k$ and after every usage of the Lemma the produced drawing with bubbles is strongly monotone. This iterative approach is used to ensure that, when defining bubbles for some vertex w_i, the previously added bubbles for $w_1, \ldots, w_{i-1}$ are taken into account.

Lemma 5 *Let Γ be a strongly monotone drawing with bubbles and let w be a vertex of degree 2 with an obtuse angle such that the two incident edges $e_1 = (u, w)$ and $e_2 = (v, w)$ have no bubbles. Then, there exist bubbles B_{e_1} and B_{e_2} for edges e_1 and e_2 respectively that only allow obtuse angles such that Γ remains strongly monotone with bubbles if we add B_{e_1} and B_{e_2}.*

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