Connected Dominating Set in Unit-Disk Graphs is W[1]-hard*

Mark de Berg† Hans Bodlaender‡† Sándor Kisfaludi-Bak†

Abstract

We prove that connected dominating set is W[1]-hard for unit-disk graphs.

1 Introduction

Wireless networks give rise to a host of interesting algorithmic problems. In the traditional model of a wireless network each node u corresponds to a disk D_u in the plane, whose radius equals the transmission range of u. Thus u can send a message to another node v if and only if $v \in D_u$. If each node has the same transmission range and we shrink each disk by a factor two, this condition is equivalent to requiring that the (shrunk) disks D_u and D_v intersect. Thus the communication graph is the intersection graph of a collection of congruent disks or, in other words, a *unit-disk graph (UDG)*. Because of their relation to wireless networks, UDGs have been studied extensively.

Let $\mathcal{D}$ be a set of disks in the plane, and let $\mathcal{G}_\mathcal{D} = (\mathcal{D}, E)$ be the UDG induced by $\mathcal{D}$. A *broadcast tree* is a rooted spanning tree for $\mathcal{G}$. To send a message from the root of the broadcast tree to all other nodes, each internal node of the tree has to send the message to its children. Hence, the cost of broadcasting is related to the number of internal nodes in the broadcast tree. A cheapest broadcast tree thus corresponds to a minimum-size *connected dominating set* on $\mathcal{G}_\mathcal{D}$, that is, a minimum-size subset $\Delta \subset \mathcal{D}$ such that the subgraph induced by Δ is connected and each node in $\mathcal{G}_\mathcal{D}$ is either in Δ or it is a neighbor of a node in Δ. Thus we are interested in the following problem: given a set $\mathcal{D}$ of n disks and a parameter k, does $\mathcal{G}_\mathcal{D}$ admit a connected dominating set of size at most k?

In the following we denote the dominating-set problem by DS, the connected dominating-set problem by CDS, and we denote these problems on UDGs by DS-UDG and CDS-UDG, respectively. It is well known that DS and CDS are NP-hard, even for planar graphs [5]. DS-UDG and CDS-UDG are also NP-hard [6, 8]. In this paper we are interested in the parameterized complexity [4] of these problems, with k being the parameter.

*This research was supported by the Netherlands' Organisation for Scientific Research (NWO) under project no. 024.002.003.

†Department of Computer Science, TU Eindhoven. mdberg@win.tue.nl, s.kisfaludi.bak@tue.nl

‡Department of Information and Computing Sciences, Utrecht University, h.l.bodlaender@uu.nl

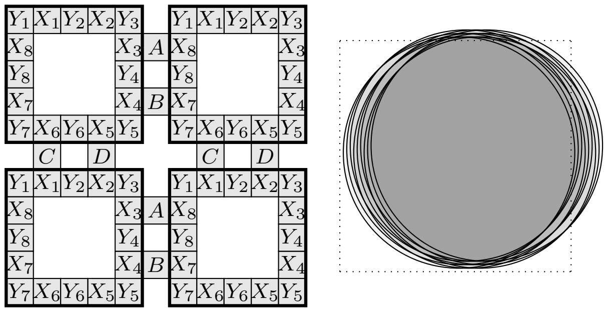

Figure 1: Left: the construction of Marx for $k = 2$. Right: example of disks inside a single block (X_2).

For general graphs DS and CDS are well-known W[2]-complete problems, but for planar graphs both problems are fixed-parameter tractable [1, 3]. The question is what happens for unit-disk graphs, which are in between general graphs and planar graphs. Marx [7] showed that DS-UDG is W[1]-hard; in this paper we extend his construction to show that CDS-UDG is W[1]-hard as well. The membership in W[1] remains open for both DS-UDG and CDS-UDG.

2 The construction by Marx for DS-UDG

Our W[1]-hardness proof for CDS-UDG has the same global structure as the W[1]-hardness proof of Marx [7] for DS-UDG. Hence, we first describe his proof. He uses a reduction from GRID TILING [2] (although Marx does not explicitly state it this way). In a grid-tiling problem we are given an integer k, an integer n, and a collection $\mathcal{S}$ of k^2 non-empty sets $S_{a,b} \subseteq [n] \times [n]$ ($1 \leqslant a, b \leqslant k$), and the goal is to select an element $s_{a,b} \in S_{a,b}$ for each $1 \leqslant a, b \leqslant k$ such that

- If $s_{a,b} = (x, y)$ and $s_{a+1,b} = (x', y')$, then $x = x'$.
- If $s_{a,b} = (x, y)$ and $s_{a,b+1} = (x', x')$, then $y = y'$.

One can picture these sets in a $k \times k$ matrix: in each cell (a, b), we need to select a representative from the set $S_{a,b}$ so that the representatives selected from horizontally neighboring cells agree in the first coordinate, and representatives from vertically neighboring sets agree in the second coordinate.

The reduction places k^2 gadgets, one for each $S_{a,b}$. A gadget contains sixteen blocks, labeled $X_1, Y_1, X_2, Y_2, \ldots, X_8, Y_8$, that are arranged in a grid.

Initially, each block X_ℓ contains n^2 disks, denoted by $X_\ell(1), \ldots, X_\ell(n^2)$ and each block Y_ℓ contains n^2+1 disks denoted by $Y_\ell(0), \ldots, Y_\ell(n^2)$. The argument j of $X_\ell(j)$ can be thought of as a pair (x, y) with $1 \leqslant x, y \leqslant n$ for which $f(x,y) = (x-1)n + y = j$. Let $f^{-1}(j) = (\iota_1(j), \iota_2(j)) = (1 + \lfloor j/n \rfloor, 1 + (j \mod n))$.

For the final construction, in each gadget at position (a, b), delete all disks $X_\ell(j)$ for each $\ell = 1, \ldots, 8$ and $(\iota_1(j), \iota_2(j)) \notin S_{a,b}$. This deletion ensures that the gadgets represent the corresponding set $S_{a,b}$. (The disks of a minimum dominating set in the gadget (a, b) will signify a specific choice $s_{a,b} = (x, y)$.)

Moreover, there are special connector blocks (denoted by A, B, C and D) between neighboring gadgets, each of them containing $n+1$ disks. A picture of the construction for $k = 3$ can be seen in Figure 1, where each block is represented by a square.

In every block, the place of each disk center is defined with regard to the midpoint of the block, (r, s). The center of each circle is of the form $(r + \alpha\epsilon, s + \beta\epsilon)$ where r, s, α and β are integers, and $\epsilon > 0$ a small constant. We say that the *offset* of the disk centered at $(r + \alpha\epsilon, s + \beta\epsilon)$ is (α, β). Note that $|\alpha|, |\beta| \leqslant n$, and $\epsilon < n^{-2}$, so the disks in a block all intersect each other. The disks of a block can be thought of as slightly shifted versions of the inscribed disk of the square in Figure 1. The exact offsets for each disk are defined in [7]. We only describe the important properties. First, two disks can intersect only if they are in the same or in neighboring blocks. Consequently, one needs at least 8 disks to dominate a gadget. The second important property is that disk $X_\ell(j)$ dominates exactly $Y_\ell(j), \ldots, Y_\ell(n^2)$ from the "previous" block Y_ℓ, and $Y_{\ell+1}(0), \ldots, Y_{\ell+1}(j-1)$ from the "next" block $Y_{\ell+1}$). This property can be used to prove the following key lemma.

Lemma 1 *Assume that a gadget is part of an instance such that none of the blocks Y_i are intersected by disks outside the gadget. If there is a dominating set Δ of the instance that contains exactly $8k^2$ disks, then there is a* canonical *dominating set Δ' with $|\Delta'| = |\Delta|$, such that for each gadget G, there is an integer $1 \leqslant j^G \leqslant n$ such that Δ' contains exactly the disks $X_1(j^G), \ldots, X_8(j^G)$ from G.*

In the gadget $G_{a,b}$, the value j defined in the above lemma represents the choice of $s_{a,b} = (\iota_1(j), \iota_2(j))$ in the grid tiling problem. Our deletion of certain disks in X-blocks ensures that $(\iota_1(j), \iota_2(j)) \in S_{a,b}$. Finally, in order to get a feasible grid tiling, gadgets in the same row must agree on the first coordinate, and gadgets in the same column must agree on the second coordinate. This depends on the following lemma.

Lemma 2 *Let Δ be a canonical dominating set. For horizontally neighboring gadgets G and H representing j_G and j_H, the disks of the connector block A are dominated if and only if $\iota_1(j_G) \leqslant \iota_1(j_H)$; the disks of B are dominated if and only if $\iota_1(j_G) \geqslant \iota_1(j_H)$. Similarly, for vertically neighboring blocks G' and H', the disks of block C are dominated if and only if $\iota_2(j_{G'}) \leqslant \iota_2(j_{H'})$; the disks of D are dominated if and only if $\iota_2(j_{G'}) \geqslant \iota_2(j_{H'})$.*

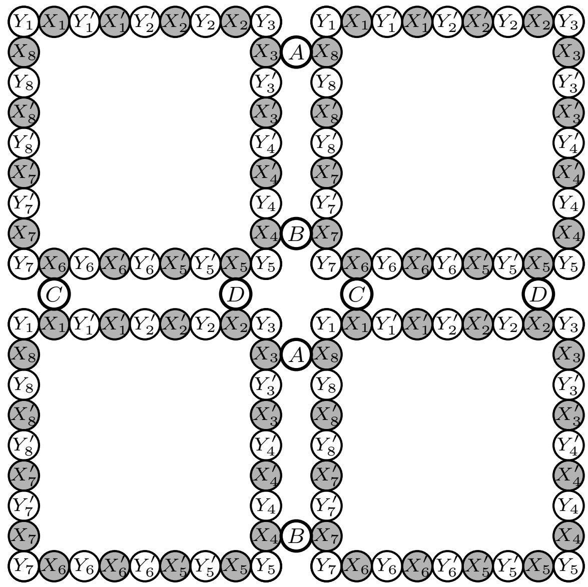

Figure 2: Connecting neighbouring blocks

With the above lemmas, it is easy to see how the reduction works. A feasible grid tiling defines a dominating set of size $8k^2$: in gadget $G_{a,b}$, the dominating disks are $X_\ell\left(f(s_{a,b})\right)$, $\ell = 1, \ldots, 8$. On the other hand, if there is a dominating set of size $8k^2$, then there is a canonical dominating set of the same size that defines a feasible grid tiling.

3 New construction for CDS-UDG

To extend the construction to CDS-UDG, we want to make sure that minimum-size dominating set is connected. This requires two things. First, we must add new disks "inside" the gadgets — that is, in the empty space surrounded by the X and Y-blocks — such that a canonical minimum dominating set includes some new disks that connect the chosen $X_\ell(j)$ disks without interfering with disks in the Y-blocks. Second, we need to connect all the different gadgets. This time in addition to avoiding the Y-blocks, we also need to avoid interfering with the connector blocks.

In order to have enough space, our gadgets contain 32 blocks instead of 16. The offsets of disks inside the blocks are not modified: we use the same building blocks. Figure 2 shows how we arrange these blocks, and depicts the connector-block placement.

The analogue of Lemma 1 and Lemma 2 are true here; we have a construction that could be used to

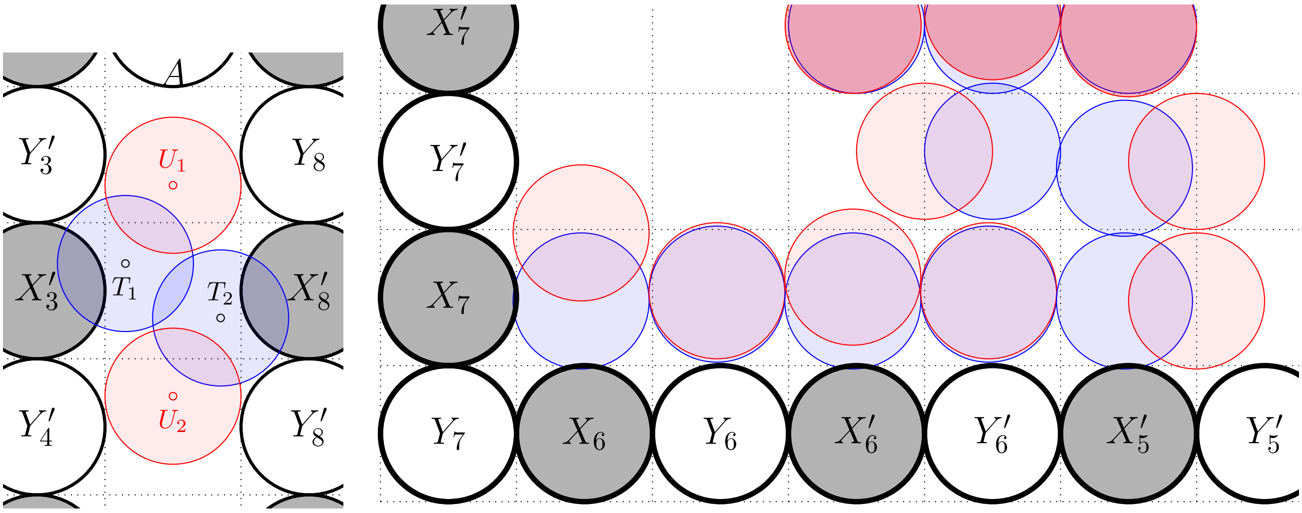

Figure 3: Left: connecting horizontally; right: connecting one side to the middle.

prove the W[1]-hardness of DS-UDG, with canonical sets of size $16k^2$, that contain one disk from each X-block and X'-block. We extend this construction so that we have canonical dominating sets that span a connected subgraph.

The disks we add are always in pairs. One of these disks (the *parent*) "connects" some other disks, or more specifically, the set of parents together with one arbitrary disk from each X and X' block is a connected subgraph of our construction. The other disk (called *leaf*) only intersects its parent and it is disjoint from all other disks. Let Δ be a dominating set. In Δ, at least one of the parent or the leaf has to be included so that the leaf is dominated. Hence, we can assume that a minimum size dominating set contains all parent disks, which (as we will ensure) form a connected set.

The most important property of the blocks that we use is that for a small enough value ϵ, the boundaries of the disks in a block all lie inside a small width annulus - for this reason, the blocks in our pictures are depicted with thick boundary disks. In order for a parent disk p to intersect every disk in a block it is sufficient if the boundary of p crosses this annulus.

We are going to add 72 extra disks to every gadget, and 4 "connector" disks between every pair of horizontally or vertically neighboring gadgets, resulting in canonical dominating sets of size $16k^2+36k^2+4k(k-1) = 56k^2-4k$ (Note that only the parent disks are included in the canonical set). In other words, the new construction has a connected dominating set of size $56k^2-4k$ if and only if there is a feasible grid tiling. Due to length constraints we will not be able to list the coordinates of these disks and prove all the intersections/disjointness that is required. These details will be available in the final version of this paper.

Connecting neighboring gadgets. For a pair of horizontally neighboring gadgets, we add two pairs of disks that connect X_3' from the left gadget to X_8' in the right gadget. This arrangement is depicted on the left of Figure 3. The parent disk with center T_1 intersects every disk in the block X_3' of the left gadget, and the other parent intersects every disk in the block X_8'. The two leaf disks (red disks in the figure) only intersect their parent. We use a rotated version of these 4 disks for vertical connections, where the parents connect X_5' from the upper gadget and X_2' from the lower gadget.

Disks inside gadgets. We begin by adding 8 disk pairs to the center. The parents are arranged in a square, touching the neighbors, and the leafs are placed so that it is possible to connect from the outside on each side. See the middle of Figure 4 for a picture: the corresponding leaf disks have a darker shade of red.

In order to connect the X-blocks, we need to connect the blocks of each side to the central disks. For this purpose, we are going to use a zigzag pattern of disks. The first parent disk intersect all disks in X_6 and X_7 (i.e., it crosses both annuli), the second parent is above the block Y_6, but it is disjoint from it. The next with center P_3 intersects all disks in X_6' , and the disk around P_4 is disjoint from the disks in Y_6'. Finally, the disk around P_5 intersects all disks in X_5'. The leafs follow a more complicated pattern. This pattern is depicted on the right side of Figure 3. Our final gadget can be attained by rotating the above seven disk pairs around the center $(8, 8)$ by $90, 180$ and 270 degrees: see Figure 4. We added the spanned edges of a canonical dominating set to this picture. This concludes the proof of our main theorem.

Theorem 3 *The* CDS-UDG *problem is W[1]-hard.*

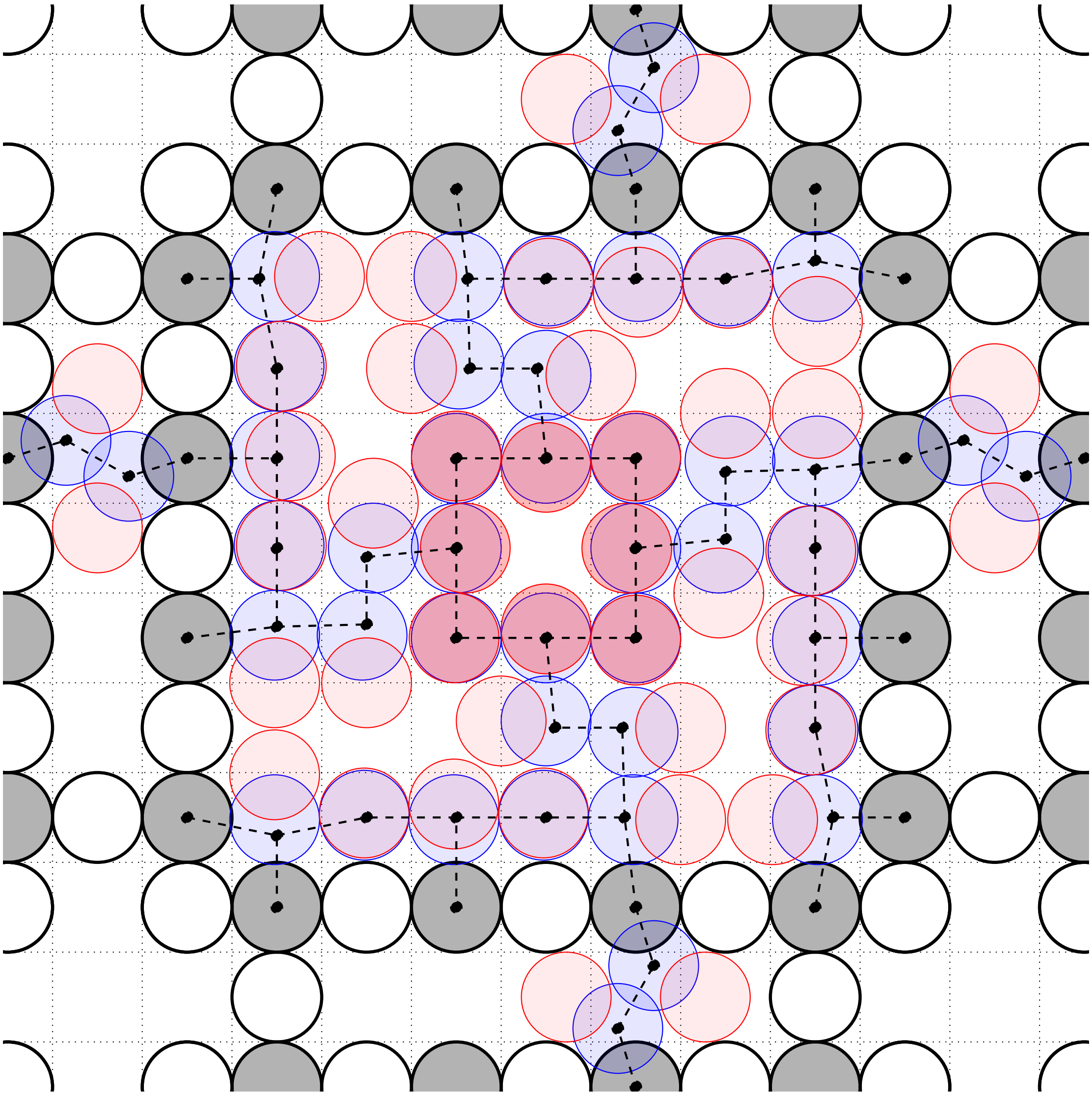

Figure 4: A gadget in the final construction. The dashed lines are spanned edges of a canonical dominating set.

References

[1] J. Alber, H. L. Bodlaender, H. Fernau, T. Kloks, and R. Niedermeier. Fixed parameter algorithms for DOMINATING SET and related problems on planar graphs. *Algorithmica*, 33(4):461–493, 2002.

[2] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.

[3] E. D. Demaine and M. Hajiaghayi. The bidimensionality theory and its algorithmic applications. *Comput. J.*, 51(3):292–302, 2008.

[4] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

[5] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

[6] D. Lichtenstein. Planar formulae and their uses. *SIAM J. Comput.*, 11(2):329–343, 1982.

[7] D. Marx. Parameterized complexity of independence and domination on geometric graphs. In *Parameterized and Exact Computation, Second International Workshop, IWPEC 2006, Zürich, Switzerland*, pages 154–165, 2006.

[8] S. Masuyama, T. Ibaraki, and T. Hasegawa. The computational complexity of the m-center problems on the plane. *IEICE TRANSACTIONS (1976-1990)*, 64(2):57–64, 1981.