Ramsey-type theorems for lines in 3-space

Jean Cardinal* Michael S. Payne† Noam Solomon‡

Abstract

We prove geometric Ramsey-type statements on collections of lines in 3-space. These statements give guarantees on the size of a clique or an independent set in (hyper)graphs induced by incidence relations between lines, points, and reguli in 3-space. Among other things, we prove the following:

- The intersection graph of n lines in $\mathbb{R}^3$ has a clique or independent set of size $\Omega(n^{1/3})$.
- Every set of n lines in $\mathbb{R}^3$ has a subset of $\sqrt{n}$ lines that are all stabbed by one line, or a subset of $\Omega\left((n/\log n)^{1/5}\right)$ lines such that no 6-subset is stabbed by one line.
- Every set of n lines in general position in $\mathbb{R}^3$ has a subset of $\Omega(n^{2/3})$ lines that all lie on a regulus, or a subset of $\Omega(n^{1/3})$ lines such that no 4-subset is contained in a regulus.

The proofs of these statements all follow from geometric incidence bounds – such as the Guth-Katz bound on point-line incidences in $\mathbb{R}^3$ – combined with Turán-type results on independent sets in sparse graphs and hypergraphs. As an intermediate step towards the third result, we also show that for a fixed family of plane algebraic curves with s degrees of freedom, every set of n points in the plane has a subset of $\Omega(n^{1-1/s})$ points incident to a single curve, or a subset of $\Omega(n^{1/s})$ points such that at most s of them lie on a curve. Although similar Ramsey-type statements can be proved using existing generic algebraic frameworks, the lower bounds we get are much larger than what can be obtained with these methods. The proofs directly yield polynomial-time algorithms for finding subsets of the claimed size.

1 Introduction

Ramsey theory studies the conditions under which particular discrete structures must contain certain substructures. Ramsey's Theorem says that for every n, every sufficiently large graph has either a clique or an independent set of size n. Early geometric Ramsey-type statements include the Happy Ending Problem on convex quadrilaterals in planar point sets, and the Erdős-Szekeres Theorem on subsets in convex position [7].

We prove a number of Ramsey-type statements involving lines in $\mathbb{R}^3$. Our proofs combine two main ingredients: geometric information in the form of bounds on the number of incidences among the objects, and a Turán-type theorem that converts this information into a Ramsey-type statement.

Ramsey's Theorem for graphs and hypergraphs only guarantees the existence of rather small cliques or independent sets. However for the geometric relations we study the bounds are known to be much larger. Therefore we are interested in finding the correct asymptotics. In particular, we are interested in the *Erdős-Hajnal property.* A class of graphs has this property if each member with n vertices has either a clique or an independent set of size n^δ for some constant $\delta > 0$. The results presented here make use of important recent advances in combinatorial geometry, a key example of which is the bound on the number of incidences between points and lines in $\mathbb{R}^3$ given by Guth and Katz [10] in their recent solution of the Erdős distinct distances problem.

1.1 A general framework

In general we consider two classes of geometric objects $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^d$ and a binary incidence relation contained in $\mathcal{P} \times \mathcal{Q}$. For a finite set $P \subseteq \mathcal{P}$ and an integer $t \geq 2$, we say that a t-subset $S \in \binom{P}{t}$ is *degenerate* whenever there exists $q \in \mathcal{Q}$ such that every $p \in S$ is incident to q. Hence the incidence relation together with the integer t induces a t-uniform hypergraph $H = (P, E)$, where $E \subseteq \binom{P}{t}$ is the set of all degenerate t-subsets of P. A clique in this hypergraph is a subset $S \subseteq P$ such that $\binom{S}{t} \subseteq E$. Similarly, an independent set is a subset $S \subseteq P$ such that $\binom{S}{t} \cap E = \emptyset$.

In what follows, the families $\mathcal{P}$ and $\mathcal{Q}$ will mostly consist of lines or points in 3-space. We are interested in Erdős-Hajnal properties for the t-uniform hypergraph H.

1.2 Previous results

When $\mathcal{P}$ is a set of points, finding a large independent set amounts to finding a large subset of points in

*Université libre de Bruxelles (ULB), jcardin@ulb.ac.be
†University of Melbourne, michael.payne@unimelb.edu.au
‡Tel Aviv University, noamsolomon@post.tau.ac.il

some kind of general position defined with respect to $\mathcal{Q}$. When $\mathcal{Q}$ is the set of points, we are dealing with intersections between the objects in $\mathcal{P}$. In particular, the case $t = 2$ corresponds to the study of geometric intersection graphs.

A set in $\mathbb{R}^d$ is usually said to be in general position whenever no $d + 1$ points lie on a hyperplane. For points and lines in the plane, Payne and Wood proved that the Erdős-Hajnal property essentially holds with exponent 1/2 [16]. Cardinal et al. proved an analogous result in $\mathbb{R}^d$ [3].

Theorem 1 ([16, 3]) *Fix $d \geq 2$. Every set of n points in $\mathbb{R}^d$ contains $\sqrt{n}$ cohyperplanar points or $\Omega((n/\log n)^{1/d})$ points in general position.*

In both cases, the proofs rely on incidence bounds, in particular the Szemerédi-Trotter Theorem [19] in the plane, and the point-hyperplane incidence bounds due to Elekes and Tóth [6] in $\mathbb{R}^d$. We streamlined the technique used in those proofs in order to easily apply it to other incidence relations.

A survey of Erdős-Hajnal properties for geometric intersection graphs was produced by Fox and Pach [8]. A general Ramsey-type statement for the case where $\mathcal{P}$ is the set of plane convex sets was proved by Larman et al. [14] more than 20 years ago. They showed that any family of n such sets contained at least $n^{1/5}$ members that are either pairwise disjoint or pairwise intersecting. Larman et al. also showed that there exist arrangements of $k^{2.3219}$ line segments with at most k pairwise crossing and at most k pairwise disjoint segments. This lower bound was improved successively by Károlyi et al. [12], and Kyncl [13].

More recently Fox and Pach studied intersection graphs of a large variety of other geometric objects [9]. In particular, they proved the Erdős-Hajnal property for families of s-intersecting curves in the plane – families such that no two curves cross more than s times. Erdős-Hajnal properties for hypergraphs have been proved by Conlon, Fox, and Sudakov [5].

A very general version of the problem for the case $t = 2$ has been studied by Alon et al. [1]. Here Ramsey-type results are provided for intersection relations between semialgebraic sets of constant description complexity in $\mathbb{R}^d$. It was shown that intersection graphs of such objects always have the Erdős-Hajnal property. The proof combines a linearisation technique with a space decomposition theorem due to Yao and Yao [20]. As an example, Alon et al. applied their machinery to prove that every family of n pairwise skew lines in $\mathbb{R}^3$ contains at least $k \geq n^{1/6}$ elements $\ell_1, \ell_2, \ldots, \ell_k$ such that ℓ_i passes above ℓ_j for all $i < j$. For the problems we consider, however, the exponents we obtain are significantly larger than what can be obtained from this method.

A more general version of this problem for arbitrary values of t has recently been studied by Conlon et al. [4], for which the Ramsey numbers grow like towers of height $t - 1$.

1.3 Our results

Section 2 deals with the case where $\mathcal{P}$ and $\mathcal{Q}$ are lines and points in $\mathbb{R}^3$. A natural object to consider is the intersection graph of lines in $\mathbb{R}^3$, for which we prove the Erdős-Hajnal property with exponent 1/3. This makes use of the Guth-Katz incidence bound between points and lines in $\mathbb{R}^3$ [11].

Section 3 deals with the setting where both $\mathcal{P}$ and $\mathcal{Q}$ are lines in $\mathbb{R}^3$. We prove that every set of n lines in $\mathbb{R}^3$ has a subset of $\sqrt{n}$ lines that are all stabbed by one line, or a subset of $\Omega\left((n/\log n)^{1/5}\right)$ such that no 6-subset is stabbed by one line. The proof involves the Ramsey-type result on points and hyperplanes due to Cardinal et al. [3], which in turn relies on a point-hyperplane incidence bound due to Elekes and Tóth [6].

Finally, in Section 4 we introduce the notion of a subset of lines in general position in $\mathbb{R}^3$ with respect to reguli, defined as loci of lines intersecting three pairwise skew lines. This uses the Pach-Sharir bound on incidences between points and curves in the plane [15].

The large subsets whose existence our results guarantee can be found in polynomial time.

Omitted proofs are given in a long version of the paper[1].

2 Points and lines in $\mathbb{R}^3$

We consider the setting in which the family $\mathcal{P}$ is the set of lines in $\mathbb{R}^3$ and $\mathcal{Q} = \mathbb{R}^3$. The first subcase we consider is $t = 2$, or in other words, intersection graphs of lines.

Theorem 2 *The intersection graph of n lines in $\mathbb{R}^3$ has a clique or independent set of size $\Omega(n^{1/3})$.*

We now sketch the proof, that combines Turán's Theorem with the Guth-Katz bounds [11, Theorem 4.5] and [11, Theorem 2.11]. The latter can be shown to yield the following.

Lemma 3 *Given a set L of n lines, so that no plane or regulus contains more than s lines, and no point is incident to more than ℓ lines of L, the number of line-line incidences is $O(n^{3/2} \log \ell + ns + n\ell)$.*

(We recall that $\omega(G)$ and $\alpha(G)$ denote the clique and independence number of a graph G, respectively.)

Lemma 4 *Consider a set L of n lines in $\mathbb{R}^3$, such that no plane contains more than s lines, and no point is incident to more than ℓ lines of L. Let G be the*

[1] http://arxiv.org/abs/1512.03236

intersection graph of L. If $s, \ell \lesssim n^{1/2}$, then $\alpha(G) \gtrsim \sqrt{n}/\log \ell$. Moreover, if $r := \max\{s, \ell\} \gtrsim n^{\frac{1}{2}+\epsilon}$ for some $\epsilon > 0$, then $\alpha(G) \gtrsim n/r$.

Proof. If there is some regulus containing at least $n^{1/2}$ lines, we divide the lines into the two rulings of the regulus. One ruling contains at least half the lines, hence $\alpha(G) \gtrsim n^{1/2}$. We may therefore assume that the number of lines contained in a common regulus is at most $n^{1/2}$.

If $s, \ell \leq n^{1/2}$, the first term in the bound in Lemma 3 dominates, and applying Turán's Theorem gives $\alpha(G) \gtrsim \sqrt{n}/\log \ell$. If $r \geq n^{\frac{1}{2}+\epsilon}$, one of the latter terms dominates, and we apply Turán's Theorem to get $\alpha(G) \gtrsim n/r$. ◻

Proof. [Theorem 2] Suppose that such a graph G has $\alpha(G) \ll n^{1/3}$. Then by Lemma 4, $\max\{s, \ell\} \gtrsim n^{2/3}$. If $\ell \gtrsim n^{2/3}$ we are done, so $s \gtrsim n^{2/3}$. Therefore, we may assume that there is a plane containing $n^{2/3}$ lines. Divide these lines into classes of pairwise parallel lines. If some class contains at least $n^{1/3}$ lines, we have $\alpha(G) \gtrsim n^{1/3}$. Otherwise, there are at least $n^{1/3}$ distinct classes. Choosing one line from each class yields a clique of size $n^{1/3}$. ◻

Note that the Erdős-Hajnal property for intersection graphs of lines in $\mathbb{R}^3$ can be directly established from Alon et al. [1], but with a much smaller exponent. For $t = 3$, we also obtain a three-dimensional version of the dual of the result of Payne and Wood (Theorem 1 with $d = 2$).

Theorem 5 *Consider a collection L of n lines in $\mathbb{R}^3$, such that at most s lie in a plane, with $s \leq n/\log n$. Then there exists a point incident to $\sqrt{n}$ lines, or a subset of $\Omega(\sqrt{n})$ lines such that at most two intersect in one point.*

3 Stabbing lines in $\mathbb{R}^3$

Three lines in $\mathbb{R}^3$ are typically intersected by a fourth line, except in certain degenerate cases. Thus it makes sense to study configurations of lines in $\mathbb{R}^3$, and to consider a set of 4 or more lines degenerate if all its elements are intersected by another line. Here we provide a result for 6-tuples of lines.

We define a 6-tuple of lines to be degenerate if all six lines are intersected (or "stabbed") by a single line in $\mathbb{R}^3$. We call this line a *stabbing line* for the 6-tuple of lines. So in our framework this is the setting in which both $\mathcal{P}$ and $\mathcal{Q}$ are the set of lines in $\mathbb{R}^3$, and $t = 6$.

We make use of the Plücker coordinates and coefficients for lines in $\mathbb{R}^3$, which are a common tool for dealing with incidences between lines, see e.g. Sharir [17]. We prove the following Ramsey-type result for stabbing lines in $\mathbb{R}^3$.

Theorem 6 *Let L be a set of n lines in $\mathbb{R}^3$. Then either there is a subset of $\sqrt{n}$ lines of L that are all stabbed by one line, or there is a subset of $\Omega\left((n/\log n)^{1/5}\right)$ lines of L such that no 6-subset is stabbed by one line.*

Theorem 6 is an immediate consequence of the following generalisation of Theorem 1. The difference is that the set of hyperplanes $\mathcal{H}$ is arbitrary instead of being the set of all hyperplanes in $\mathbb{R}^d$. The proof is similar to that of Cardinal et al. [3].

Theorem 7 *Let $\mathcal{H}$ be a set of hyperplanes in $\mathbb{R}^d$. Then, every set of n points in $\mathbb{R}^d$ with at most ℓ points on any hyperplane in $\mathcal{H}$, where $\ell = O(n^{1/2})$, contains a subset of $\Omega\left((n/\log \ell)^{1/d}\right)$ points so that every hyperplane in $\mathcal{H}$ contains at most d of these points.*

We also have a simple construction for the following upper bound.

Theorem 8 *For every constant integer $t \geq 4$, there exists an arrangement L of n lines in $\mathbb{R}^3$ such that there is no subset of more than $O(\sqrt{n})$ lines that are all stabbed by one line, nor any subset of more than $O(\sqrt{n})$ lines with no t stabbed by one line.*

4 Lines and reguli in $\mathbb{R}^3$

Consider the case in which $\mathcal{P}$ is the class of lines in $\mathbb{R}^3$, $\mathcal{Q}$ is the class of reguli, and $t = 4$. Let P be a set of n lines, and assume that the lines in P are pairwise skew. Every triple of lines in P therefore determines a single regulus, and we may consider the set of all reguli determined by P. We consider the containment relation rather than intersection – we are interested in 4-tuples that all lie in the same regulus.

In order to prove our result, we first consider the case where $\mathcal{P} = \mathbb{R}^2$ and $\mathcal{Q}$ is a family of algebraic curves of bounded degree. We define the number of degrees of freedom of a family of algebraic curves $\mathcal{C}$ to be the minimum value s such that for any s points in $\mathbb{R}^2$ there are a constant number of curves passing through all of them. Moreover, $\mathcal{C}$ has multiplicity type r if any two curves in $\mathcal{C}$ intersect in at most r points. The proof of the following uses the Pach-Sharir bounds on the number of incidences between points and curves [15].

Theorem 9 *Consider a family $\mathcal{C}$ of bounded degree algebraic curves in $\mathbb{R}^2$ with constant multiplicity type and s degrees of freedom, for some $s > 2$. Then in any set of n points in $\mathbb{R}^2$, there exists a subset of $\Omega(n^{1-1/s})$ points incident to a single curve of $\mathcal{C}$, or a subset of $\Omega(n^{1/s})$ points such that at most s of them lie on a curve of $\mathcal{C}$.*

We now come back to our original question in which $\mathcal{P}$ is the class of lines in $\mathbb{R}^3$, $\mathcal{Q}$ is the class of reguli, and $t = 4$. Here we restrict the finite arrangement $P \subset \mathcal{P}$ to be pairwise skew, that is, pairwise nonintersecting and nonparallel. Recall that a regulus can be defined as a quadratic ruled surface which is the locus of all lines that are incident to three pairwise skew lines. There are only two kinds of reguli, both of which are quadrics – hyperbolic paraboloids and hyperboloids of one sheet [18].

Theorem 10 *Let L be a set of n pairwise skew lines in $\mathbb{R}^3$. Then there are $\Omega(n^{2/3})$ lines on a regulus, or $\Omega(n^{1/3})$ lines, no 4-subset of which lies on a regulus.*

The bounds can be shown to be tight.

Theorem 11 *There exists a set P of n pairwise skew lines in $\mathbb{R}^3$ such that there is no subset of more than $O(n^{2/3})$ lines on a regulus, and no more than $O(n^{1/3})$ lines such that no 4-subset lies on a regulus.*

Note that Aronov et al. [2] proved an upper bound on the number of incidences between lines and reguli in 3-space, from which one may derive an alternative proof of Theorem 10.

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