Coloring and $L(2,1)$-labeling of unit disk intersection graphs

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Abstract

In this paper we give a family of on-line algorithms for the classical coloring problem and the $L(2,1)$-labeling of unit disc intersection graphs. Our algorithms make use of a geometric representation of such graphs and are inspired by an algorithm of Fiala *et al.*, but have better competitive ratios. The improvement comes from an application of a fractional and a b-fold coloring of the plane. Moreover, we give an off-line algorithm improving the bound of the $L(2,1)$-span of unit disk intersection graphs in terms of the maximum degree.

1 Introduction

Intersection graphs of families of geometric objects attracted much attention of researches both for their theoretical properties and practical applications (c.f. McKee and McMorris [10]). For example intersection graphs of families of discs, and in particular discs of unit diameter (called *unit disk intersection graphs*), play a crucial role in modeling radio networks. Apart from the classical coloring, other labeling schemes such as T-coloring and distance-constrained labeling of such graphs are applied to frequency assignment in radio networks [9, 13].

In this paper we consider the classical coloring and the $L(2,1)$-labeling. The latter asks for a vertex labeling with non-negative integers, such that adjacent vertices get labels that differ by at least two, and vertices at distance two get different labels. The *span* of an $L(2,1)$-labeling is the maximum label used. The $L(2,1)$-*span* of a graph G, denoted by $\lambda(G)$, is the minimum span of an $L(2,1)$-labeling of G (note that the number of available labels is $\lambda(G)+1$, but some may not be used).

We say that a graph coloring algorithm is *on-line* if the input graph is not known a priori, but is given vertex by vertex (with all edges adjacent to already revealed vertices). Each vertex is colored at the moment when it is presented and its color cannot be changed later. On the other hand, *off-line* coloring algorithms know the whole graph before they start assigning colors. The on-line coloring can be much harder than off-line coloring, even for paths. For an off-line coloring algorithm (off-line $L(2,1)$-labeling algorithm, resp.), by the *approximation ratio* we mean the worst-case ratio of the number of colors used by this algorithm (the largest label used by this algorithm, resp.) to the chromatic number of the graph ($\lambda(G)$, respectively). For on-line algorithms, the same value is called the *competitive ratio*.

A unit disk intersection graph G can be colored off-line in polynomial time with $3\omega(G)$ colors [12] (where $\omega(G)$ denotes the size of a maximum clique) and on-line with $5\omega(G)$ colors [11, 12]. Fiala *et al.* [3] presented an on-line algorithm that finds an $L(2,1)$-labeling of a unit disk intersection graph with span not exceeding $25\omega(G)$. The algorithm is based on a special pre-coloring of the plane, that resembles colorings studied by Exoo [2], inspired by the classical Hadwiger-Nelson problem [8]. Our main result are on-line algorithms for the coloring and the $L(2,1)$-labeling of unit disc intersection graphs with better competitive ratios than previous algorithms. They are inspired by [3], although a b-fold coloring of the plane (see [7]) is used instead of a classical coloring. In particular, in the case of using 1-fold coloring we obtain the algorithm from [3]. Our algorithm colors (in the classical sense) unit disc intersection graphs with large maximum clique, using less than $5\omega(G)$ colors and hence it is the best currently known approximation on-line coloring algorithm for such graphs. For $L(2,1)$-labeling, in the case of 1-fold coloring of the plane, our algorithm gives a labeling with span not exceeding $20\omega(G)$. Using b-fold coloring for $b>1$ we obtain even better results.

For general graphs, Griggs and Yeh proved that $\lambda(G) \le \Delta(G)^2 + 2\Delta(G)$ and conjectured that $\lambda(G) \le \Delta(G)^2$. Shao *et al.*[14] showed $\lambda(G) \le \frac{4}{5}\Delta(G)^2 + 2\Delta(G)$ if $G \in UDG$. Actually, they gave an on-line algorithm that finds an $L(2,1)$-labeling of G with span at most $\frac{4}{5}\Delta(G)^2 + 2\Delta(G)$. We managed to improve this bound to $\frac{3}{4}\Delta^2 + 3(\Delta-1)$, in the off-line case. Moreover, we show that the algorithm from [3] implies the bound $18\Delta + 18$, which is better for $\Delta \ge 22$.

Throughout the paper we always assume that the input unit disk intersection graph is given along with its geometric representation. In practical application for mobile Wi-Fi routers representation can be found

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with methods from [5].

2 Preliminaries

For an integer n, we define $[n] := \{1, \dots, n\}$. A function $c\colon V \to [k]$ is a k-coloring of $G = (V, E)$ if for any $xy \in E$ holds $c(x) \neq c(y)$. By $d(u, v)$ we denote the number of edges on the shortest u-v–path in G.

For a sequence of unit discs in the plane $(D_i)_{i\in[n]}$ we define its intersection graph by $G((D_i)_{i\in[n]}) = (\{v_i : i \in [n]\}, E)$, where v_i is the center of D_i for every $i \in [n]$ and $v_iv_j \in E$ iff $D_{v_i} \cap D_{v_j} \neq \emptyset$. Notice that $v_iv_j \in E$ if and only if the Euclidean distance between v_i and v_j, denoted by $\mathrm{dist}(v_1, v_2)$, is at most one. By *UDG* we mean the class of graphs that admit a representation by intersecting unit disks.

For a minimization on-line algorithm alg, by cr(alg) we denote its *competitive radio*, which is the supremum of $\frac{\mathrm{alg}(G)}{\mathrm{opt}(G)}$ over all instances G, where $\mathrm{alg}(G)$ is the value of the solution given by the algorithm for instance G and $\mathrm{opt}(G)$ is the optimal solution for instance G. For the classical coloring we use fact that any coloring requires at least $\omega(G)$ colors, where $\omega(G)$ denotes the size of the largest clique of G. By $\mathcal{G}_\omega$ we denote the class of graphs with largest clique of size at least ω and by $\mathrm{cr}(\mathrm{alg}(\mathcal{G}_\omega))$ we denote the supremum of $\frac{\mathrm{alg}(G)}{\mathrm{opt}(G)}$ over all graphs $G \in \mathcal{G}_\omega$.

A *tiling* is a partition of the plane into convex polygons with partially removed boundary, called *tiles*, such that every two points from one tile are at distance less than one. If we have b tilings, then by a *subtile* we mean a non-empty intersection of b tiles, one from each tiling. We will use a hexagon as a tile and hexagon tiling, just as Fiala *et al.* [3].

A function $\varphi : \mathbb{R}^2 \to [k]$ is called a *coloring of the plane with the color set* $[k]$ if for any two points $p_1, p_2 \in \mathbb{R}^2$ with $\mathrm{dist}(p_1, p_2) = 1$ holds $\varphi(p_1) \neq \varphi(p_2)$.

Definition 1 *A function $\varphi = (\varphi_1, \dots, \varphi_b)$ where $\varphi_i : \mathbb{R}^2 \to [k]$ for $i \in [b]$ is called a* b-fold coloring of the plane with color set $[k]$ *if*

- *for any point $p \in \mathbb{R}^2$ and $i, j \in [b]$, if $i \neq j$, then $\varphi_i(p) \neq \varphi_j(p)$,*
- *for any two points $p_1, p_2 \in \mathbb{R}^2$ with $\mathrm{dist}(p_1, p_2) = 1$ and $i, j \in [b]$ holds $\varphi_i(p_1) \neq \varphi_j(p_2)$.*

The function φ_i for $i \in [b]$ is called an i-th layer *of φ.*

Notice that a coloring of the plane is a 1-fold coloring of the plane. A coloring of a plane φ is called tiling-based if there exists a tiling such that each tile is monochromatic and adjacent tiles have different colors. A b-fold coloring of a plane $\varphi = (\varphi_1, \dots, \varphi_b)$ is called tiling-based if for every $i \in [b]$ coloring φ_i is tiling-based.

For technical reasons, we shall consider $L(2, 1)$-labelings with labels starting with one. To avoid confusion, we shall call such labelings $L(2, 1)$-*colorings*. Formally, a k-$L(2, 1)$-coloring of a graph G is any function $c : V \to [k]$ such that

1. $|c(v) - c(w)| \geq 1$ for all $v, w \in V(G)$ such that $d(u, w) = 2$,
2. $|c(v) - c(w)| \geq 2$ for all $v, w \in V(G)$ such that $vw \in E(G)$.

Definition 2 *A b-fold coloring of the plane φ is called a* b-fold $L^*(2, 1)$-coloring of the plane with color set $[k]$ *if for any two points $p_1, p_2 \in \mathbb{R}^2$:*

- $\mathrm{dist}(p_1, p_2) = 1 \Rightarrow \forall_{i_1,i_2\in\{1,\dots,b\}}\ 2 \leq |\varphi_{i_1}(p_1) - \varphi_{i_2}(p_2)| < k - 1$,
- $1 < \mathrm{dist}(p_1, p_2) \leq 2 \Rightarrow \forall_{i_1,i_2\in\{1,\dots,b\}}\ 1 \leq |\varphi_{i_1}(p_1) - \varphi_{i_2}(p_2)|$.

By $L^*(2, 1)$-coloring of the plane we mean 1-fold $L^*(2, 1)$-coloring of the plane.

3 On-line coloring

The main idea of the algorithm is as follows. We start with some fixed tiling-based b-fold coloring $\varphi = (\varphi_1, \dots, \varphi_b)$ of the plane with colors $[k_\varphi]$. When a disc D is read, it is assigned to one of the b layers of φ (we try to distribute discs to layers as uniformly as possible). Then a tile from this layer that contains a center of D is found. The vertex corresponding to D is colored with the color of this tile plus k_φ multiplied by the number of vertices previously assigned to this tile.

Algorithm $Color_\varphi((D_i)_{i\in[n]})$
1. ForEach $i \in [n]$
2. Read D_i, let v_i be the center of D_i
3. ForEach $r \in [b]$ let $T_r(v_i)$ be the tile from the layer r containing v_i
4. $\ell(v_i) \leftarrow 1+(|\{v_1, \dots v_{i-1}\} \cap \bigcap_{r\in[b]} T_r(v_i)| \pmod b)$
5. $t(v_i) \leftarrow |\{u \in \{v_1, \dots v_{i-1}\} \cap T_{\ell(v_i)}(v_i) : \ell(u) = \ell(v_i)\}|$
6. $c(v_i) \leftarrow \varphi_{\ell(v_i)}(v_i) + k_\varphi \cdot t(v_i)$
7. Return c

Theorem 1 *Let φ be a tiling-based b-fold coloring of the plane with color set $[k_\varphi]$, and $(D_i)_{i\in[n]}$ be a sequence of unit discs. Algorithm $Color_\varphi((D_i)_{i\in[n]})$ returns a coloring of $G := G((D_i)_{i\in[n]})$. Moreover, if φ is a b-fold $L^*(2, 1)$-coloring of the plane, then Algorithm $Color_\varphi((D_i)_{i\in[n]})$ returns an $L(2, 1)$-coloring of G.*

Theorem 2 *Let φ be a b-fold coloring of the plane with color set $[k_\varphi]$, with at most γ_φ subtiles in one* *tile, and let $(D_i)_{i\in[n]}$ be a sequence of unit discs. Algorithm $Color_\varphi(\ (D_i)_{i\in[n]})$ returns coloring of $G := G((D_i)_{i\in[n]})$ with the highest color not exceeding $k_\varphi \cdot \lfloor \frac{\omega(G)+(b-1)\gamma_\varphi}{b} \rfloor$.*

Proof. Let $\gamma = \gamma_\varphi$ and let v_i be a vertex that got the biggest color. Consider the moment of the course of the algorithm when vertex v_i was colored. Let $\ell(v_i), t(v_i), c(v_i)$ be defined as in the algorithm. Let $T = T_{\ell(v_i)}$ be the tile from the $\ell(v_i)$-th layer containing v_i. Let $S_1, \dots S_\gamma$ be the subtiles of T. Let $s_q = |\{u \in \{v_1, \dots v_i\} : u \in S_q\}|$ and $s_q(\ell(v_i)) = |\{u \in \{v_1, \dots v_i\} : u \in S_q, \ell(u) = \ell(v_i)\}|$ for $q \in [\gamma]$.

Notice that, thanks to the formula in line 4 of the algorithm, vertices in the subtile $\bigcap_{r\in[b]} T_r(v_i)$ are almost uniformly distributed among layers.

The key observation is that by the definition of $\ell(v_i)$ we get $s_q \geq b\cdot(s_q(\ell(v_i)) - 1) + \ell(v_i)$. Now we are ready to estimate the number of vertices from $\{v_1, \dots, v_i\}$ contained in $T_{\ell(v_i)}$. Notice that these vertices are pairwise at distance less than one and hence they form a clique. We obtain

$$\begin{aligned}\omega(G) \geq \sum_{q=1}^{\gamma} s_q \geq \sum_{q=1}^{\gamma} b\cdot(s_q(\ell(v_i)) - 1) + \ell(v_i)\\ \geq b\cdot\left[\sum_{q=1}^{\gamma}(s_q(\ell(v_i)) - 1)\right] + \gamma\\ = b\cdot(t(v_i)+1) - (b-1)\gamma\end{aligned}$$

and thus

$$t(v_i) \leq \left\lfloor \frac{\omega(G)+(b-1)\gamma}{b} - 1 \right\rfloor.$$

Finally we obtain $c(v_i) \leq k_\varphi \cdot \lfloor \frac{\omega(G)+(b-1)\gamma}{b} \rfloor$ which, by the choice of v_i, is the highest color used. □

This shows that is it crucial to construct good b-fold colorings of the plane. We are able to do this if b is a square number.

Theorem 3 *For $h \in \mathbb{N}_+$ there exists a tiling-based h^2-fold coloring of the plane with $\left\lceil (\frac{2}{\sqrt{3}}+1)\cdot h \right\rceil^2$ colors and $\gamma_\varphi = 6h^2$.*

Directly from Theorems 2 and 3 we obtain:

Corollary 4 *For the h^2-fold φ coloring of the plane from Theorem 3 we have*

$$\begin{aligned}&\mathrm{cr}(Color_\varphi(\mathcal{G}_\omega)) \leq\\ &\frac{\left\lceil (\frac{2}{\sqrt{3}}+1)\cdot h \right\rceil^2}{\omega}\cdot\left\lfloor \frac{\omega + (h^2-1)6h^2}{h^2} \right\rfloor\\ &= 4.65 + O\left(\frac{1}{h}\right) + O\left(\frac{h^4}{\omega}\right).\end{aligned}$$

Notice that for $h = 5$ and graphs G with $\omega(G) \geq 108901$, the competitive ratio of the algorithm is less than 5.

Analogously to Theorem 3, we are able to construct a good b-fold $L^*(2,1)$-coloring of the plane.

Theorem 5 *There exists b-fold tiling-based $L^*(2,1)$-coloring φ of the plane for*

1. *$b = 1$ with color set $[20]$ (see Figure 1),*
2. *$b = 2$ with color set $[34]$ and the parameter $\gamma_\varphi = 4$ (see Figure 2),*
3. *$b = 3$ with color set $[49]$ and the parameter $\gamma_\varphi = 6$,*
4. *$b = h^2$ for $h \in \mathbb{N}$ with $3\rho^2 + 1$ colors, where $\rho = \left\lceil h(\frac{2}{\sqrt{3}}+1)+1 \right\rceil$, and $\gamma_\varphi = 6h^2$*

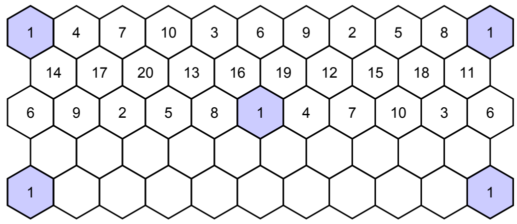

Figure 1: 1-fold $L^*(2,1)$-coloring of the plane

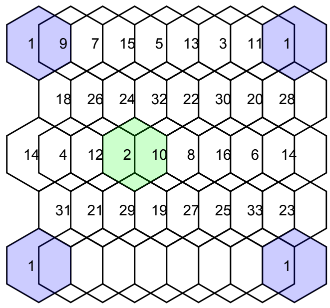

Figure 2: 2-fold $L^*(2,1)$-coloring of the plane

Corollary 6 *For $b \in \mathbb{N}$ and b-fold $L^*(2,1)$-colorings φ of the plane from Theorem 5, the value $\mathrm{cr}(Color_\varphi(\mathcal{G}_\omega))$ is at most:*

1. *$10 + \frac{10}{2\omega-1}$ for φ from Theorem 5.1,*
2. *$8.5 + \frac{76.5}{2\omega-1}$ for φ from Theorem 5.2,*
3. *$8\frac{1}{6} + \frac{204.17}{2\omega-1}$ for φ from Theorem 5.3,*
4. *$(3\lceil h(\frac{2}{\sqrt{3}}+1)+1\rceil^2+1)\frac{\omega+6h^2(h^2-1)}{h^2(2\omega-1)}$ $= 6.97 + O\left(\frac{1}{h}\right) + O\left(\frac{h^4}{\omega}\right)$ for φ from Theorem 5.4.*

4 Off-line $L(2,1)$-labeling

In this section we give an improvement for the following theorem by Shao *et al.* [14], which partially answers the question of Calamoneri [1, Section 4.7.1].

Theorem 7 (Shao *et al.* [14]) *If $G \in UDG$, then $\lambda(G) \leq \frac{4}{5}\Delta^2 + 2\Delta$.*

By Δ we denote the maximum degree of the input graph G. Fix some vertex v. By V_L we denote the half-plane lying left of v (including the boundary). A neighbor w of v is a *left neighbor* if $w \in V_L$. A neighbor w of v is *important*, if it is a left neighbor of v, or w has a neighbor $w' \in V_L$, such that $\text{dist}(w', u) > 1$ for every left neighbor u of v (in particular, w' is not a neighbor of v). It is easy to verify that each vertex v has at most 3 pairwise non-adjacent left neighbors. The following lemma is the strengthening of this observation.

Lemma 8 *Let $G \in UDG$. Each vertex has at most 4 pairwise non-adjacent important neighbors.*

Now we can present the first bound.

Lemma 9 *Let $G \in UDG$ and $\Delta \geq 7$. Then $\lambda(G) \leq \frac{3}{4}\Delta^2 + 3(\Delta - 1)$.*

Proof. We use a greedy algorithm, processing vertices from left to right. Consider a vertex v. By N_1 we denote the set of the left neighbors of v, and by N_2 we denote the set of important right neighbors of v and by N^2 we denote the set of vertices left of v, which are not in N_1, but share a common neighbor with v. Observe that our algorithm will never use a label bigger than $3|N_1|+|N^2|$. Let $d := |N_1 \cup N_2| \leq \Delta$.

Suppose that $N_2 \neq \emptyset$. Let $H = G[N_1 \cup N_2]$. By Lemma 8, it does not contain an independent set of size 5, so its complement, $\bar{H}$, is K_5-free. By the famous theorem of Turán, the maximum number of edges in $\bar{H}$ is $\frac{3}{4}\frac{d^2}{2}$. So the number of edges in H is at least $\binom{d}{2} - \frac{3}{4}\frac{d^2}{2} = \frac{d}{2}\left(\frac{d}{4} - 1\right)$. This gives us the following: $|N^2| \leq d(\Delta - 1) - 2 \cdot \frac{d}{2}\left(\frac{d}{4} - 1\right) \leq \frac{3}{4}\Delta^2$. Since $|N_2| > 0$ and thus $|N_1| \leq \Delta - 1$, we obtain that $3|N_1| + |N^2| \leq 3(\Delta - 1) + \frac{3}{4}\Delta^2 = \frac{3}{4}\Delta^2 + 3\Delta - 3$.

It is easy to verify that if $N_2 = \emptyset$, then $3|N_1| + |N^2| \leq \frac{2}{3}\Delta^2 + 3\Delta < \frac{3}{4}\Delta^2 + 3\Delta - 3$ for $\Delta \geq 7$. $\square$

Fiala *et al.* [3] proved that if $G \in UDG$, then $\lambda(G) \leq 18\omega(G)$. Since $\omega(G) \leq \Delta + 1$, we obtain the following corollary.

Corollary 10 *Let G be a unit disk graph with maximum degree at most Δ. Then $\lambda(G) \leq 18\Delta + 18$.*

Combining the bound $\lambda(G) \leq \Delta^2 + 2\Delta - 2$ by Gonçalves [4], the bound from Lemma 9 and the bound from Corollary 10, we get the following.

Theorem 11 *If $G \in UDG$, then $\lambda(G) \leq f(\Delta)$ for*

$$f(\Delta) = \begin{cases} \Delta^2 + 2\Delta - 2 & \text{if } \Delta < 7, \\ \frac{3}{4}\Delta^2 + 3\Delta - 3 & \text{if } 7 \leq \Delta < 22, \\ 18\Delta + 18 & \text{if } \Delta \geq 22. \end{cases}$$

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