Stabbing circles for some sets of Delaunay segments

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Abstract

Let S be a set of n disjoint segments in the plane that correspond to edges of the Delaunay triangulation of some fixed point set. Our goal is to compute all the combinatorially different stabbing circles for S, and the ones with maximum and minimum radius. We exploit a recent result to solve this problem in $O(n \log n)$ time in two cases: (i) all segments in S are parallel; (ii) all segments in S have the same length. We also show that the problem of computing the stabbing circle of minimum radius of a set of n parallel segments of equal length (not necessarily edges of a Delaunay triangulation) has an $\Omega(n \log n)$ lower bound.

1 Introduction

The *stabbing circle problem* is formulated as follows: Let S be a set of n segments in $\mathbb{R}^2$ in general position (segments have $2n$ distinct endpoints, no three endpoints are collinear, and no four of them are cocircular). A circle c is a stabbing circle for S if exactly one endpoint of each segment of S is contained in the exterior of the closed disk induced by c; see Fig. 1. The stabbing circle problem consists of (1) reporting a representation of all the combinatorially different stabbing circles for S (two circles are *combinatorially different* if the sets of endpoints in the exterior of the corresponding disks are different); and (2) finding stabbing circles with minimum and maximum radius.

Figure 1: Left: Segment set with a stabbing circle. Right: Segment set with no stabbing circle.

The stabbing circle problem has antecedents in the stabbing line problem, which was solved in optimal $\Theta(n \log n)$ time by Edelsbrunner et al. [6]. Other stabbing shapes (wedges, isothetic rectangles, etc.) have also been considered; see [4] for an overview.

The problem of stabbing a set S of n segments in the plane by a circle can be solved in $O(n^2)$ time by a combination of known results, and this is worst-case optimal [4]. Recently, we presented an alternative algorithm based on connecting the problem to *cluster Voronoi diagrams* [4]. We identified conditions under which the algorithm is subquadratic; these conditions are: (1) the Hausdorff Voronoi diagram and the farthest-color Voronoi diagram have linear structural complexity and can be constructed in subquadratic time (see Section 2 for the definition of these diagrams); (2) a technical condition related to the number of times an edge of the Hausdorff Voronoi diagram contains centers of combinatorially different stabbing circles. If the segments in S are parallel, conditions (1) and (2) are satisfied, and the stabbing circle problem for S can be solved in $O(n \log^2 n)$ time.

In this note we continue investigating special instances of segment sets for which the algorithm in [4] is subquadratic, in order to understand the stabbing circle problem better. We focus on sets S of disjoint segments that correspond to edges in the Delaunay triangulation of a fixed point set. We solve the stabbing circle problem in $O(n \log n)$ time when all segments in S are either parallel or have the same length. We also show an $\Omega(n \log n)$ lower bound for the problem of computing the stabbing circle of minimum radius of a set of n parallel segments of equal length (not necessarily edges of a Delaunay triangulation).

2 Preliminaries

In what follows, xx' denotes either a segment in S, or the pair of its endpoints as convenient.

Definition 1 [5, 9] *The* Hausdorff Voronoi diagram *of S is a partitioning of $\mathbb{R}^2$ into the following regions:*

$$\mathsf{hreg}(aa') = \{p \in \mathbb{R}^2 \mid \forall bb' \in S \setminus \{aa'\}: \max\{d(p,a), d(p,a')\} < \max\{d(p,b), d(p,b')\}\};$$
$$\mathsf{hreg}(a) = \{p \in \mathsf{hreg}(aa') \mid d(p,a) > d(p,a')\}.$$

The *graph structure* of this diagram is $\mathsf{HVD}(S) = \mathbb{R}^2 \setminus \bigcup_{aa' \in S} (\mathsf{hreg}(a) \cup \mathsf{hreg}(a'))$. An edge of $\mathsf{HVD}(S)$ is *pure* if it is incident to regions of two distinct segments.

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Definition 2 [1, 7] *The* farthest-color Voronoi diagram *is a partitioning of* $\mathbb{R}^2$ *into the following regions:*

$$\mathsf{fcreg}(aa') = \{p \in \mathbb{R}^2 \mid \forall bb' \in S \setminus \{aa'\}: \min\{d(p,a), d(p,a')\} > \min\{d(p,b), d(p,b')\}\};$$
$$\mathsf{fcreg}(a) = \{p \in \mathsf{fcreg}(aa') \mid d(p,a) < d(p,a')\}.$$

The graph structure of this diagram is $\mathsf{FCVD}(S) = \mathbb{R}^2 \setminus \bigcup_{aa' \in S} (\mathsf{fcreg}(a) \cup \mathsf{fcreg}(a'))$.

For arbitrary segments, the combinatorial complexity of both diagrams is $O(n^2)$ [9, 1]. If the segments are disjoint, the complexity of $\mathsf{HVD}(S)$ is O(n) [5].

Let $\overline{\mathsf{hreg}}(\cdot)$ and $\overline{\mathsf{fcreg}}(\cdot)$ denote the closures of the respective Voronoi regions.

Definition 3 *Given a point* p*, the* Hausdorff disk *of* p*, denoted* $D_h(p)$*, is the closed disk centered at* p *of radius* $d(p,a)$*, where* $p \in \overline{\mathsf{hreg}}(a)$*.*

Let S be a set of n pairwise disjoint segments in $\mathbb{R}^2$ in general position; let S have no stabbing line. In [4] we presented an algorithm to solve the stabbing circle problem for S. To state it, we need some notation.

Let e be a pure edge of $\mathsf{HVD}(S)$ and let w be a point in e. In [4] we defined a set *type*(w), whose elements might be $\tilde{l}$, $\tilde{r}$, *mm*, *in*, and *out*. The meaning of *type*(w) is not essential for this note; it is enough to point out that *type*(w) can be found in $O(1)$ time if w is located in $\mathsf{FCVD}(S)$.

The *find-change query* is defined as follows: Given two points t, s in e such that *type*(t) contains $\tilde{r}$ but not $\tilde{l}$, and *type*(s) contains $\tilde{l}$ but not $\tilde{r}$, the query returns a point w in the segment ts such that one of the following holds: (i) $\{\tilde{r}, \tilde{l}\} \subseteq$ *type*(w); (ii) *in* $\in$ *type*(w); (iii) *out* $\in$ *type*(w).

Suppose that $e = uv$ is a portion of the border of $\mathsf{hreg}(a)$ and $\mathsf{hreg}(b)$, for $aa', bb' \in S$. We say that a segment $cc' \in S \setminus \{aa', bb'\}$ is of type *middle* for e if either c or c' is contained in $D_h(u) \setminus D_h(v)$ and the other endpoint in $D_h(v) \setminus D_h(u)$.

Let m denote the number of pairs formed by a segment $cc' \in S$ and a pure edge e of $\mathsf{HVD}(S)$ such that cc' is of type *middle* for e. We build the results of Section 3 of this abstract on the following result.

Theorem 1 [4] *The stabbing circle problem for* S *can be solved in* $O(\mathcal{T}_{\mathsf{HVD}(S)} + \mathcal{T}_{\mathsf{FCVD}(S)} + |\mathsf{HVD}(S)|\mathcal{T}_{fc} + |\mathsf{FCVD}(S)|\log n + m\mathcal{T}_{fc})$ *time, where* $\mathcal{T}_{\mathsf{HVD}(S)}$ *(resp.,* $\mathcal{T}_{\mathsf{FCVD}(S)}$*) is the time to compute* $\mathsf{HVD}(S)$ *(resp.,* $\mathsf{FCVD}(S)$*),* $|\mathsf{HVD}(S)|$ *(resp.,* $|\mathsf{FCVD}(S)|$*) is the number of edges of* $\mathsf{HVD}(S)$ *(resp.,* $\mathsf{FCVD}(S)$*), and* $\mathcal{T}_{fc}$ *is the time to answer a find-change query.*

3 Segments with the Delaunay property

We say that S *satisfies the Delaunay property* if its segments correspond to edges of some Delaunay triangulation. Let us assume that S satisfies this property.

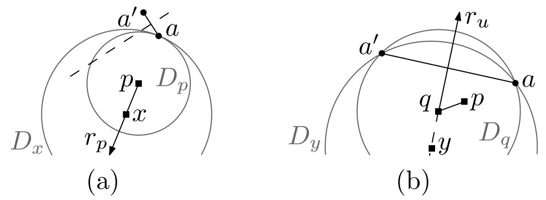

Figure 2: (a) $r_p \cap bis(a,a') = \emptyset$ and $D_p \subset D_x$. (b) $r_p \cap bis(a,a') = \{q\}$ and $D_{q\ell} \subset D_y$.

Lemma 2 $\mathsf{FCVD}(S)$ *is a tree of* $O(n)$ *complexity.*

Proof. We show that $\mathsf{FCVD}(S)$ for such a segment set S is an instance of the *farthest abstract Voronoi diagram* (FAVD); the claim then follows automatically from [8]. To prove that $\mathsf{FCVD}(S)$ is FAVD, we consider the *nearest-color* Voronoi diagram of S, which reveals the nearest site (segment in S), where the distance from a point $p \in \mathbb{R}^2$ to some $aa' \in S$ is $\min\{d(p,a), d(p,a')\}$. We need to prove that the system of bisectors for farthest/nearest color Voronoi diagram satisfies the following axioms: (1) each bisector is an unbounded Jordan curve; (2) any two bisectors intersect finite number of times; (3) regions of the nearest-color Voronoi diagram are (a) non-empty, (b) path-connected, and (c) cover $\mathbb{R}^2$. Note that the nearest-color Voronoi diagram is related to the nearest-point Voronoi diagram of all endpoints of S: the region of $aa' \in S$ in the former diagram is the union of the regions of a and a' in the latter.

Our bisector system satisfies axioms (2), (3a) and (3c) since so does the bisector system of the nearest/farthest point Voronoi diagram. Further, since each $aa' \in S$ is an edge of the Delaunay triangulation of all endpoints of S, the regions of a and a' in the nearest-point Voronoi diagram are adjacent, thus their union is path-connected, implying axiom (3b). A bisector in our system satisfies axiom (1), since it separates two unions of pairs of adjacent regions in the diagram of four points. ☐

The faces of $\mathsf{FCVD}(S)$ near infinity coincide with the faces of the farthest-segment Voronoi diagram of S, thus, their sequence at infinity can be computed in $O(n \log n)$ time by divide and conquer (and other methods) [10]. Based on this observation, it is simple to derive a divide and conquer algorithm for $\mathsf{FCVD}(S)$. (Note that the approach in [8] yields an expected $O(n \log n)$ time algorithm for $\mathsf{FCVD}(S)$.)

Lemma 3 $\mathsf{FCVD}(S)$ *can be constructed in* $O(n \log n)$ *time and* $O(n)$ *space.*

Let $bis(a,b)$ denote the bisector of a and b.

Lemma 4 *For a point* $p \in \mathbb{R}^2$*, let* r_p *be the open ray with origin at* p *and direction* $\overrightarrow{ap}$*, where* a *is the endpoint of* $aa' \in S$ *such that* $p \in \overline{\mathsf{fcreg}}(a)$*. Let*

$p \notin bis(a,a')$. If $r_p \cap bis(a,a') = \{q\}$, then $\mathsf{fcreg}(aa')$ contains the open segment pq, as well as one of the two (unbounded) portions of $bis(a,a')$ starting at q. Otherwise, $r_p \subset \mathsf{fcreg}(aa')$.

Proof. For any point $z \in \mathbb{R}^2$, let D_z be the disk centered at z of radius $d(z,a)$; see Fig. 2.

Suppose that r_p does not intersect $bis(a,a')$. Since $p \in \overline{\mathsf{fcreg}}(a)$, disk D_p contains an endpoint of every segment in S. For a point $x \in r_p, x \neq p$, $D_p \subset D_x$. Thus D_x contains in its interior an endpoint of every segment in $S \setminus \{aa'\}$, that is, $x \in \mathsf{fcreg}(a) \subseteq \mathsf{fcreg}(aa')$.

Suppose next that r_p intersects $bis(a,a')$ in a point q. For any point $x \in pq$, $x \neq p$, we have $x \in \mathsf{fcreg}(a) \subset \mathsf{fcreg}(aa')$ by the above argument. In particular, disk D_q contains an endpoint of every segment in S. Point q breaks $bis(a,a')$ into two rays r_u and r_ℓ, which are respectively above and below q (see Fig. 2b), and aa' breaks disk D_q into two parts D_{qu} and $D_{q\ell}$ that are above and below aa' respectively. (We assume that aa' is not vertical, otherwise the above/below relation can be replaced by left/right.) Observe that, if $\mathsf{fcreg}(aa')$ does not contain r_u (resp., r_ℓ), then $D_{q\ell}$ (resp., D_{qu}) contains an endpoint of some segment in $S \setminus \{aa'\}$. If $\mathsf{fcreg}(aa')$ contained neither r_u nor r_ℓ, there would be an endpoint of a segment in $S \setminus \{aa'\}$ inside $D_{q\ell}$, and an endpoint inside D_{qu}. A contradiction to aa' being an edge of the Delaunay triangulation of the set of endpoints of S. ☐

Lemma 5 *$\mathsf{FCVD}(S)$ can be preprocessed in $O(n \log n)$ time and $O(n)$ space so that a find-change query is answered in $O(\log n)$ time.*

Proof. By Lemma 2 $\mathsf{FCVD}(S)$ is a tree, and thus the *centroid decomposition* [3] can be built for it, and used to answer the find-change query. This decomposition is a (graph-theoretical) balanced tree with n nodes, one for each vertex of $\mathsf{FCVD}(S)$, built in $O(n \log n)$ time by finding the *centroid* vertex c of the tree $\mathsf{FCVD}(S)$, making it a root, and recursing into the three connected components of $\mathsf{FCVD}(S) \setminus \{c\}$. The subtree of each node v corresponds to a connected portion of $\mathsf{FCVD}(S)$, adjacent to the vertex v. To perform a query, we follow a root-to-leaf path (of length $O(\log n)$) in this balanced tree, at every node of the path one of the node's three subtrees is to be chosen.

We can make a decision related to one node in $O(1)$ time, thus answering a find-change query in $O(\log n)$ time. Indeed, Lemma 4 if applied to v and each of the three regions of $\mathsf{FCVD}(S)$ incident to v, induces a decomposition of $\mathbb{R}^2$ into three regions of $O(1)$ combinatorial complexity, each of which contains one subtree of v in $\mathsf{FCVD}(S)$, see Fig. 3a. Out of these three regions, in constant time we choose the only one that may contain the answer to the find-change query. ☐

We next bound the parameter m in Theorem 1.

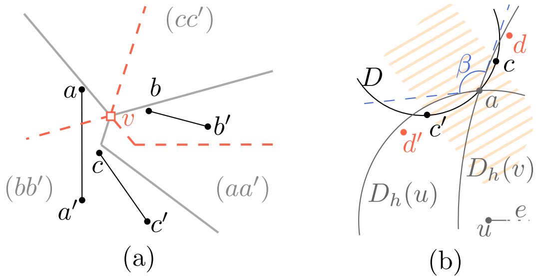

Figure 3: (a) $S = \{aa', bb', cc'\}$ (black); $\mathsf{FCVD}(S)$ (gray); the decomposition of $\mathbb{R}^2$ induced by its vertex v (red, dashed); (b) Figure for the proof of Lemma 7.

Consider a pure edge $e = uv$ of $\mathsf{HVD}(S)$ separating $\mathsf{hreg}(a)$ and $\mathsf{hreg}(b)$, for two segments $aa', bb' \in S$. Then $e \subseteq bis(a,b)$. We assume that segment ab is vertical with a on top of b, and that ab does not intersect the interior of e (otherwise e could be broken into two parts, considered separately). For any segment $cc' \in S$, we denote its supporting line by $\ell(cc')$.

Lemma 6 *If $cc' \in S$ is of type middle for S, then $\ell(cc')$ lies either above both aa', bb' or below them.*

Proof. One endpoint of cc' is in $D_h(u) \setminus D_h(v)$, and the other in $D_h(v) \setminus D_h(u)$. These two areas are separated by the vertical line $\ell(ab)$, so cc' is not vertical.

We first prove that it is impossible that a, b, c, c' are in convex position with c and c' not consecutive along the convex hull of the four points. Assume otherwise. The center of the circle through a, b and c lies on e; hence c' is outside this circle. Thus a and b are adjacent in the Delaunay triangulation of a, b, c, c'. Since this triangulation is plane, c and c' are not adjacent, and therefore they are not adjacent in the Delaunay triangulation of all endpoints of S; a contradiction.

Since c' (resp., c) is outside the circle through a, b and c (resp., c'), the convex hull of a, b, c, c' cannot be a triangle with c' (resp., c) in its interior. Hence, a and b are on the same side of $\ell(cc')$. Recall that a' and b' lie in $D_h(u) \cap D_h(v)$. Segment cc' either does not intersect $D_h(u) \cap D_h(v)$, or it divides $D_h(u) \cap D_h(v)$ in two portions, and both a, b lie in one of them. In both cases, the claim follows. ☐

Lemma 7 *If S satisfies the Delaunay property and all segments in S are of the same length, then an edge e of $\mathsf{HVD}(S)$ has at most two segments of type middle.*

Proof. We show that there is at most one segment of type middle whose supporting line is above aa', bb'. Then the claim follows from Lemma 6.

Suppose for contradiction that cc', dd' are segments of type middle for e such that $\ell(cc')$ and $\ell(dd')$ lie above aa', bb'. A vertical ray shot from a hits both cc' and dd'. Assume that it hits cc' first. Let D denote the disk through c, c', a. See Fig. 3b. Since cc' is a Delaunay edge, cc' and dd' are pairwise disjoint, and a and at least one of d, d' are on opposite sides of cc', disk D contains none of d, d'.

We have $\angle dad' > \pi/2$: it is greater than the angle β formed by the two tangents to $D_h(u)$ and $D_h(v)$ at a (see blue dashed lines in Fig. 3b) and $\beta \geq \pi/2$ by our assumption that segment ab does not intersect e in its interior. Let $s(cc')$ be the closed strip formed by two lines perpendicular to $\ell(cc')$ and passing through c and c' (tiled area in Fig. 3b). We have: d, d' are outside D; d, d' are separated by $\ell(ab)$; $\angle dad' > \pi/2$; and $\ell(dd')$ lies above cc'. All this together imply that d and d' lie outside $s(cc')$ and on different sides of it. Thus $d(d, d') < d(c, c')$; a contradiction. ☐

Recall Theorem 1. By Lemma 5, $\mathcal{T}_{fc} = O(\log n)$. Both $\mathcal{T}_{\mathsf{FCVD}(S)}$ and $\mathcal{T}_{\mathsf{HVD}(S)}$ are $O(n \log n)$, see Lemma 3 and [4]. If all segments in S are parallel, then $m = O(n)$ [4]. By Lemma 7, m is also $O(n)$ if the segments in S have the same length. We conclude:

Theorem 8 *If S satisfies the Delaunay property and either all segments in S are parallel, or all segments in S are of equal length, then the stabbing circle problem can be solved in $O(n \log n)$ time and $O(n)$ space.*

4 Lower bound

We finally prove a lower bound for sets of segments possibly without the Delaunay property, but with the other two conditions considered in this note.

Theorem 9 *The problem of computing a stabbing circle of minimum radius for a set of n parallel segments of equal length has an $\Omega(n \log n)$ lower bound in the algebraic decision tree model.*

Proof. The reduction, very similar to that of Theorem 6 in [2], is from $\mathsf{MAXGAP}(X)$. In our version, the input X consists of a set of n integers $x_1, \ldots, x_n$, and $\mathsf{MAXGAP}(X)$ is the problem of finding the maximum difference between consecutive elements of X.

Without loss of generality, we may assume $\min X = 1$. Let $x'_1 < x'_2 < \cdots < x'_n$ be the sorting of the elements of X. Then $x'_1 = 1$, and let $M = x'_n$. We construct a set S of parallel segments of equal length as follows: For every $x_i \in X$, we add a segment connecting point $(x_i, 0)$ to $(-(M+1) + x_i, 0)$. Additionally, we add two segments aa' and bb' such that $a = (-1/2, 0)$, $a' = (-(M+1) - 1/2, 0)$, $b = (1/2, 0)$, and $b' = ((M+1) + 1/2, 0)$.

Any stabbing circle for S of minimum radius contains a, b in its interior. Thus the possibilities for such a stabbing circle are: If the associated disk contains $a, b, (x'_1, 0), \ldots, (x'_n, 0)$, or $(-(M+1)+x'_1, 0), \ldots, (-(M+1)+x'_n, 0), a, b$, then it has diameter $M+1/2$. If it contains $(-(M+1) + x'_{i+1}, 0), \ldots, (-(M+1) + x'_n, 0), a, b, (x'_1, 0), \ldots, (x'_i, 0)$ for $i < n$, then it has diameter $M + 1 - (x'_{i+1} - x_i)$. Since $\mathsf{MAXGAP}(X) \geq 1$, the stabbing circles of minimum radius belong to the last family. Thus $\mathsf{MAXGAP}(X)$ is equivalent to finding the stabbing circle for S of minimum radius.

The set S does not satisfy all the assumptions of this paper, since all endpoints are collinear and the segments are not pairwise disjoint. We construct a set S' obtained from S by translating every segment vertically by distinct values of at most $\varepsilon = 1/10$. Since ε is small compared to the difference between distinct values of diameters of different stabbing circles for S (which is at least $1/2$), a minimum stabbing circle for S' corresponds to a minimum stabbing circle for S which is combinatorially "the same". This proves that the lower bound also holds for the more restricted sets of segments considered in this paper. ☐

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