# 1-bend Upward Planar Drawings of SP-digraphs with the Optimal Number of Slopes*

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## Abstract

It is proved that every series-parallel digraph whose maximum vertex-degree is $\Delta$ admits an upward planar drawing with at most one bend per edge such that each segment along each edge has one of $\Delta$ distinct slopes. This is shown to be worst-case optimal in terms of the number of slopes. Furthermore, our construction gives rise to drawings with optimal angular resolution $\frac{\pi}{\Delta}$.

## 1 Introduction

The *k-bend planar slope number* of a family of planar graphs with maximum vertex-degree $\Delta$ is the minimum number of distinct slopes used for the edges when computing a crossing-free drawing with at most $k > 0$ bends per edge of any graph in the family. For example, if $\Delta = 4$, a classic result is that every planar graph has a crossing-free drawing such that every edge segment is either horizontal or vertical and each edge has at most two bends (see, e.g., [1]). Clearly this bound on the number of slopes is optimal. This result has been extended to values of $\Delta$ larger than four by Keszegh et al. [5], who prove that $\lceil \frac{\Delta}{2} \rceil$ slopes suffice to construct a planar drawing with at most two bends per edge for any planar graph.

However if additional geometric constraints are imposed on the crossing-free drawing, only a few tight bounds on the planar slope number are known. For example, if one requires that the edges cannot have bends, the best known upper bound on the planar slope number is $O(c^{\Delta})$ (for a constant $c > 1$) while a general lower bound of just $3\Delta - 6$ has been proved [5]. Tight bounds are only known for outerplanar graphs [6] and subcubic graphs [3], while the gap between upper and lower bound has been reduced for planar graphs with treewidth two [8] or three [4]. If one bend per edge is allowed, Keszegh et al. [5] show an upper bound of $2\Delta$ and a lower bound of $\frac{3}{4}(\Delta-1)$ on the planar slope number of the planar graphs with maximum vertex-degree $\Delta$. In a recent paper, Knauer and Walczak [7] lower the upper bound to $\frac{3}{2}(\Delta - 1)$; in the same paper, it is also proved that a tight bound of $\lceil \frac{\Delta}{2} \rceil$ can be achieved for outerplanar graphs.

In this paper we focus on the 1-bend planar slope number of directed graphs with the additional requirement that the computed drawings are *upward*, i.e., each edge is drawn as a curve monotonically increasing in the $y$-direction. We recall that upward drawings are a classic research topic in graph drawing, see e.g. [2]. We show that every series-parallel digraph $G$ whose maximum vertex-degree is $\Delta$ has *1-bend upward planar slope number* $\Delta$. That is, $G$ admits an upward planar drawing with at most one bend per edge where at most $\Delta$ distinct slopes are used for the edges. This is shown to be worst-case optimal in terms of the number of slopes.

To prove the above results, we first construct a suitable contact representation of a series-parallel (di)graph where each vertex is represented as a cross, i.e. a horizontal segment properly intersected by a vertical segment (Section 3); then, we transform such contact representation into a 1-bend (upward) planar drawing optimizing the number of slopes used in such transformation (Section 4). Our construction gives rise to drawings with optimal angular resolution (i.e. the minimum angle between any two consecutive edges around a vertex); namely, the angular resolution is at least $\frac{\pi}{\Delta}$. Preliminaries are in Section 2. We conclude with some open problems in Section 5.

## 2 Preliminaries

A *series-parallel digraph* (*SP-digraph* for short) [2] is a simple planar digraph that has one source and one sink, called *poles*, and it is recursively defined as follows. A single edge is an SP-digraph. The digraph obtained by identifying the sources and the sinks of two SP-digraphs is an SP-digraph (*parallel composition*). The digraph obtained by identifying the sink of one SP-digraph with the source of a second SP-digraph is an SP-digraph (*series composition*). A *reduced* SP-digraph is a SP-digraph without transitive edges. The underlying undirected graph of an SP-digraph is called an *SP-graph*. An SP-digraph $G$ is naturally associated with a binary tree $T$, which is called the *decomposition tree* of $G$. The nodes of $T$ are of three types, *Q-nodes*, *S-nodes*, and *P-nodes*, rep-

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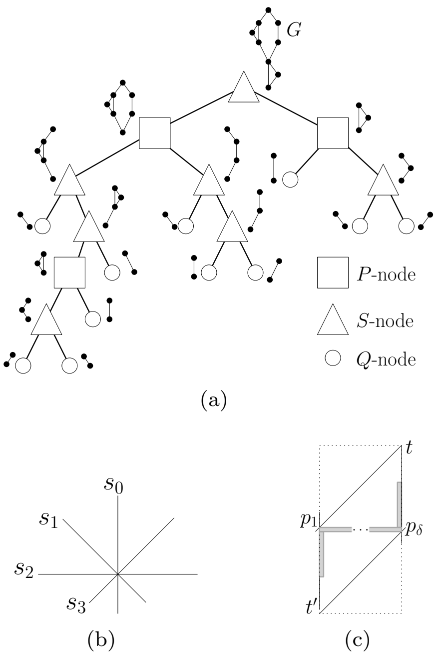



Figure 1: (a) An SP-graph $G$ and its decomposition tree. (b) The slope-set $\mathcal{S}_4$. (c) The safe-region (dotted) of a cross (when $\Delta$=4).

resenting single edges, series compositions, and parallel compositions, respectively. An SP-graph $G$ and its decomposition tree are shown in Fig. 1(a). The decomposition tree of $G$ has $O(n)$ nodes and can be constructed in $O(n)$ time [2].

The *slope* $s$ of a line $\ell$ is the angle that a horizontal line needs to be rotated counter-clockwise in order to make it overlap with $\ell$. The slope of a segment is the slope of the supporting line containing it. We denote by $\mathcal{S}_\Delta$ the set of slopes: $s_i = \frac{\pi}{2} + i\frac{\pi}{\Delta}$ $(i = 0, \ldots, \Delta - 1)$ (see Fig. 1(b)).

## 3 Cross Contact Representations

**Basic definitions.** A *cross* is composed of one horizontal segment and one vertical segment that share an interior point, the *center* of the cross. A cross is *degenerate* if either its horizontal or its vertical segment has zero length. The center of a degenerate cross is its midpoint. A point $p$ of a cross $c$ is an *end-point* (*interior point*) of $c$ if it is an end-point (interior point) of the horizontal or vertical segment of $c$. Two crosses $c_1$ and $c_2$ *touch* if they share a point $p$, called *contact*, such that $p$ is an end-point of the vertical (horizontal) segment of $c_1$ and an interior point of the horizontal (vertical) segment of $c_2$. A *cross-contact representation (CCR)* of a graph $G$, is a drawing $\gamma$ such that: (*i*) Every vertex $v$ of $G$ is represented by a (possibly degenerate) cross $c(v)$; (*ii*) All intersections of crosses are touches, and (*iii*) Two crosses $c(u)$ and $c(v)$ touch if and only if $G$ contains the edge $(u, v)$.

We now consider CCRs of digraphs, and define properties that will be useful to transform the computed CCR into a 1-bend upward drawing of the corresponding digraph with few slopes and good angular resolution. Let $\gamma$ be a CCR of a digraph $G$ with maximum vertex-degree $\Delta$. Let $(u, v)$ be an edge of $G$ oriented from $u$ to $v$. Let $p$ be the contact between $c(u)$ and $c(v)$. The point $p$ is an *upward contact* if the following two conditions hold: (a) $p$ is an end-point of the vertical segment of one of the two crosses and an interior point of the other cross, and (b) the center of $c(v)$ is above the center of $c(u)$. A CCR of a digraph $G$ such that all its contacts are upward is an *upward CCR (UCCR)*. An UCCR $\gamma$ is *balanced* if for every non-degenerate cross $c(u)$ of $\gamma$, we have that $|n_l(u) - n_r(u)| \leq 1$, where $n_l(u)$ ($n_r(u)$) is the number of contacts on the left (right) of the center of $c(u)$. Also, let $\{p_1, p_2, \ldots, p_\delta\}$ be the $\delta \geq 1$ contacts along the horizontal segment of $c(u)$, in this order from the leftmost one ($p_1$) to the rightmost one ($p_\delta$). Let $t$ be the intersection point between the vertical line passing through $p_\delta$ and the line with slope $\frac{\pi}{2} - \frac{\pi}{\Delta}$ and passing through $p_1$. Similarly, let $t'$ be the intersection point between the vertical line passing through $p_1$ and the line with slope $\frac{\pi}{2} - \frac{\pi}{\Delta}$ and passing through $p_\delta$. The *safe-region* of $c(u)$ is the rectangle having $t$ and $t'$ as the top-right and bottom-left corner, respectively. See Fig. 1(c) for an illustration. If $\delta = 1$, the safe-region degenerates to a point. An UCCR $\gamma$ is *well-spaced* if no two safe-regions intersect each other.

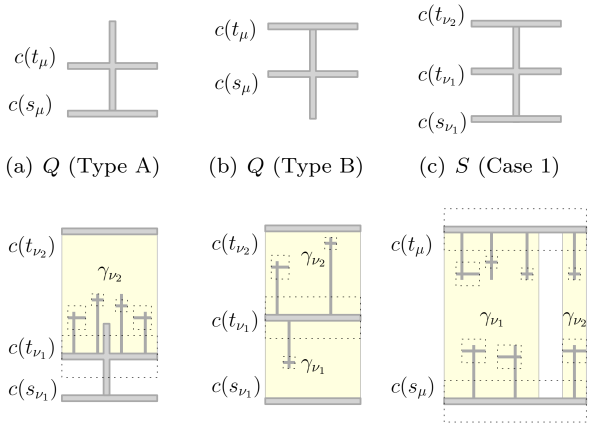
**Algorithm overview.** In the remainder of this section we describe a linear-time algorithm, `UCCRDrawer`, that takes as input a reduced SP-digraph $G$, and computes an UCCR $\gamma$ of $G$ which is balanced and well-spaced. The algorithm computes $\gamma$ through a bottom-up visit of the decomposition tree $T$ of $G$. For each node $\mu$ of $T$, `UCCRDrawer` computes an UCCR $\gamma_\mu$ of the graph $G_\mu$ associated with $\mu$ such that the following properties hold:

**P1.** $\gamma_\mu$ is balanced.

**P2.** $\gamma_\mu$ is well-spaced.

**P3.** Let $s_\mu$ and $t_\mu$ be the two poles of $G_\mu$. If $\mu$ is a $P$- or an $S$-node, then $\gamma_\mu$ is contained in a rectangle $R_\mu$ such that its bottommost (topmost) side is the cross representing $c(s_\mu)$ ($c(t_\mu)$), which is therefore degenerate.

**Drawing construction.** As already said, `UCCRDrawer` computes $\gamma$ through a bottom-up visit of the decomposition tree $T$ of $G$. For each leaf node $\mu$ (which is a $Q$-node) the associated graph $G_\mu$ consists of a single edge $(s_\mu, t_\mu)$. In this case, we define two possible types of UCCR, $\gamma^A_\mu$ (type A) and $\gamma^B_\mu$ (type B), of $G_\mu$, as in Figs. 2(a) and 2(b). Properties **P1** − **P2** trivially hold in this case, while property **P3** does not apply.




(a) $Q$ (Type A) (b) $Q$ (Type B) (c) $S$ (Case 1)

(d) $S$ (Case 2) (e) $S$ (Case 3) (f) $P$

Figure 2: Illustration for algorithm `UCCRDrawer`. The safe-regions are dotted (they are not in scale).

For each non-leaf node $\mu$ of $T$, `UCCRDrawer` computes the UCCR $\gamma_\mu$ by suitably combining the (already) computed UCCRs $\gamma_{\nu_1}$ and $\gamma_{\nu_2}$ of the two graphs associated with the children $\nu_1$ and $\nu_2$ of $\mu$. If $\mu$ is an $S$-node of $T$, we distinguish between the following cases, where we assume that $t_{\nu_1} = s_{\nu_2}$ is the pole shared by $\nu_1$ and $\nu_2$.

**Case 1.** Both $\nu_1$ and $\nu_2$ are $Q$-nodes. Then an UCCR of $G_\mu$ is computed by combining $\gamma^A_{\nu_1}$ and $\gamma^B_{\nu_2}$ as in Fig. 2(c). Properties **P1** − **P3** trivially hold.

**Case 2.** $\nu_1$ is a $Q$-node , while $\nu_2$ is not (the case when $\nu_2$ is a $Q$-node and $\nu_1$ is not, is symmetric). We combine the drawing $\gamma^A_{\nu_1}$ of $G_{\nu_1}$ and the drawing $\gamma_{\nu_2}$ of $G_{\nu_2}$ as in Fig. 2(d). Notice that, to combine the two drawings we may need to scale one of them so that their widths are the same. To ensure **P1**, we move the vertical segment $s$ of $c(t_{\nu_1}) = c(s_{\nu_2})$ so that $|n_l(t_{\nu_1}) - n_r(t_{\nu_1})| \leq 1$. We may also need to shorten its upper part in order to avoid crossings with other segments, and to extend its lower part so that $c(s_{\nu_1})$ is outside the safe-region of $c(t_{\nu_1}) = c(s_{\nu_2})$, thus guaranteeing property **P2**. Property **P3** holds by construction.

**Case 3.** If none of $\nu_1$ and $\nu_2$ is a $Q$-node, then we combine $\gamma_{\nu_1}$ and $\gamma_{\nu_2}$ as in Fig. 2(e). Also in this case we may need to scale one of the two drawings so that their widths are the same. Property **P1** holds, as it holds for $\gamma_{\nu_1}$ and $\gamma_{\nu_2}$. Property **P2** may not hold for $c(t_{\nu_1}) = c(s_{\nu_2})$. We can ensure **P2** by performing the following stretching operation. Let $\ell_a$ and $\ell_b$ be two horizontal lines slightly above and slightly below the horizontal segment of $c(t_{\nu_1}) = c(s_{\nu_2})$, respectively. We extend all the vertical segments intersected by $\ell_a$ or $\ell_b$ until the safe-region of $c(t_{\nu_1}) = c(s_{\nu_2})$ does not intersect any other safe-region. Property **P3** holds by construction.

Finally, suppose that $\mu$ is a $P$-node of $T$, having $\nu_1$ and $\nu_2$ as children (recall that neither $\nu_1$ nor $\nu_2$ is a $Q$-node, since $G$ is a reduced SP-digraph). We combine $\gamma_{\nu_1}$ and $\gamma_{\nu_2}$ as in Fig. 2(f). We may need to scale one of the two drawings so that their heights are the same. Property **P1** holds, as it holds for $\gamma_{\nu_1}$ and $\gamma_{\nu_2}$. To ensure **P2**, a stretching operation similar to the one described in Case 3 is possibly performed by using a horizontal line slightly above (below) the horizontal segment of $c(s_\mu)$ ($c(t_\mu)$). Property **P3** holds by construction.

It is possible to show that algorithm `UCCRDrawer` can be implemented to run in linear time. The above discussion can be used to prove the following.

**Lemma 1** *Let $G$ be an $n$-vertex reduced SP-digraph. Algorithm `UCCRDrawer` computes a balanced and well-spaced UCCR $\gamma$ of $G$ in $O(n)$ time.*

## 4 1-bend Upward Planar Drawings

In this section we first describe how to transform an UCCR of a reduced SP-digraph into an upward 1-bend planar drawing that uses the slopes in the slope-set $\mathcal{S}_\Delta$. We then explain how to deal with general SP-digraphs.

Let $\gamma$ be an UCCR of a reduced SP-digraph $G$ and let $c(u)$ be the cross representing a vertex $u$ of $G$ in $\gamma$. Let $p_1, \ldots, p_\delta$ ($\delta \geq 1$) be the contacts along the horizontal segment of $c(u)$. We assume that these contacts are ordered such that we first have all the contacts corresponding to the outgoing edges of $u$ from left to right, and then we have all the contacts corresponding to the incoming edges of $u$ from right to left. Let $c$ be either the center of $c(u)$, if $c(u)$ is non-degenerate, or $p_{\lfloor \frac{\delta}{2} \rfloor + 1}$ if $c(u)$ is degenerate. Consider the set of lines $\ell_0, \ldots, \ell_{\Delta-1}$, such that $\ell_i$ passes through $c$ and has slope $s_i \in \mathcal{S}_\Delta$ (for $i = 0, \ldots, \Delta - 1$). These lines intersect all the vertical segments forming a contact on the horizontal segment of $c(u)$. In particular, each quadrant of $c(u)$ contains a number of lines that is at least the number of vertical segments touching $c(u)$ in that quadrant. Since $\gamma$ is well-spaced, all these intersections are inside the safe-region of $c(u)$. Hence we can replace each contact of $c(u)$ with two segments having slope in $\mathcal{S}_\Delta$ as shown in Fig. 3(a) and 3(b). More precisely, each contact $p_i$ of $c(u)$ is replaced with two segments that are both in the quadrant of $c(u)$ that contains the vertical segment defining $p_i$. This guarantees the upwardness of the resulting drawing. Also, each edge has at most one bend. Namely, each edge is represented by a single contact between a horizontal and a vertical segment and we introduce one bend only when dealing with the cross containing the horizontal segment. Finally, $\Gamma$ is planar. Namely, there is no crossing in $\gamma$ and each cross is only modified locally inside its safe-region which, by the well-spaced property, is disjoint by any other safe-region.

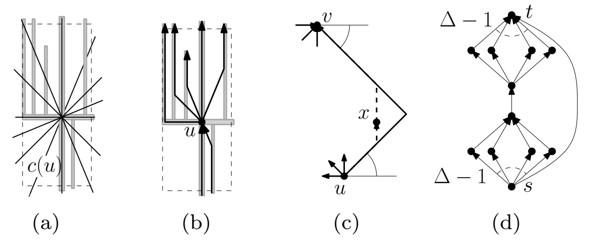



Figure 3: (a)-(b) Transforming an UCCR into a 1-bend drawing. (c) Drawing of a transitive edge. (d) A SP-digraph requiring at least $\Delta$ slopes in any 1-bend upward planar drawing.

Using the technique described above, we can compute an 1-bend upward planar drawing with slopes in the slope-set $\mathcal{S}_\Delta$ for reduced SP-digraphs. We now explain how to deal with a general SP-digraph $G$. First, we change the embedding of $G$ as follows. Let $(u, v)$ be a transitive edge, and let $G'$ be the maximal subgraph of $G$ having $u$ and $v$ as poles. We change the embedding of $G'$ so that $(u, v)$ is the rightmost outgoing edge of $u$ and the rightmost incoming edge of $v$. Second, we subdivide $(u, v)$ with a dummy vertex $x$. The resulting graph $G_r$ is a reduced SP-digraph and therefore we can compute an 1-bend upward planar drawing $\Gamma_r$ of $G_r$ as described above. When doing so, we take care of guaranteeing that the drawings of $(u, x)$ and $(x, v)$ (for each transitive edge $(u, v)$) do not use the horizontal slope (it is not hard to see that this is always possible). Each transitive edge $(u, v)$ of $G$ is represented in $\Gamma_r$ by a path of two edges $(u, x)$ and $(x, v)$. If at least one between $(u, x)$ and $(x, v)$ is drawn with no bend, then it is sufficient to remove $x$ to obtain a 1-bend drawing of $(u, v)$. If both $(u, x)$ and $(x, v)$ have one bend, then simply removing the subdivision vertex would lead to a 2-bend drawing of $(u, v)$. In this case we have to modify the drawing of $(u, v)$. Let $\ell_u$ be the straight line passing through $u$ and the bend of $(u, x)$ and let $\ell_v$ be the straight line passing through $v$ and the bend of $(x, v)$. We obtain a 1-bend drawing of $(u, v)$ by placing a single bend at the intersection point of $\ell_u$ and $\ell_v$ (see Figure 3(c)). Since we did not use the horizontal slope in the drawing of $(u, x)$ and $(x, v)$ such a point exists. With this operation, the drawing of $(u, v)$ has been extended to the right, and it is possible to modify the construction of the UCCR $\gamma$ so that $(u, v)$ does not cross any other edge. The modification of `UCCRDrawer` is such that when a $P$-node is processed, it additionally ensures the existence of an empty region where $(u, v)$ can be drawn without crossings. Details are omitted.

We conclude by exhibiting in Fig. 3(d) a family of SP-digraphs, such that, for every value of $\Delta$, there exists a graph in this family with maximum vertex-degree $\Delta$ and that requires at least $\Delta$ slopes in any 1-bend upward planar drawing. Namely, if a graph $G$ has a source (or a sink) of degree $\Delta$, then it requires at least $\Delta - 1$ slopes in any upward drawing because each slope, with the only possible exception of the horizontal one, can be used for a single edge. In the graph of Fig. 3(d) however, the edge $(s, t)$ must be either the leftmost or the rightmost edge of $s$ and $t$ in any upward drawing. Therefore, if only $\Delta - 1$ slopes are allowed, such edge cannot be drawn planarly and with one bend. Thus, the following theorem holds.

**Theorem 2** *Every $n$-vertex SP-digraph $G$ with maximum vertex-degree $\Delta$ admits a 1-bend upward planar drawing $\Gamma$ with at most $\Delta$ slopes and angular resolution at least $\frac{\pi}{\Delta}$. These bounds on the number of slopes and on the angular resolution are worst-case optimal. Also, $\Gamma$ can be computed in $O(n)$ time.*

Since every SP-graph can be oriented to an SP-digraph, next corollary is implied by Theorem 2 and lowers the upper bound for planar graphs in [7].

**Corollary 1** *The 1-bend planar slope number of SP-graphs with maximum vertex-degree $\Delta$ is at most $\Delta$.*

## 5 Conclusions and Open Problems

We proved that the 1-bend upward planar slope number of SP-digraphs with maximum vertex-degree $\Delta$ is at most $\Delta$ and this is a tight bound. Is the bound of Corollary 1 also tight? Moreover, can it be extended to any partial 2-tree?

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