Colouring Contact Graphs of Squares and Rectilinear Polygons

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Abstract

We study colourings of contact graphs of squares and rectilinear polygons. Our main results are that (i) it is NP-hard to decide if a contact graph of unit squares is 3-colourable, and (ii) any contact graph of a set of rectilinear polygons is 6-colourable.

1 Introduction

In graph-colouring problems the goal is to assign a colour to each node in a graph $\mathcal{G} = (V, E)$ such that the resulting colouring satisfies certain properties. The standard property is that for any edge $(u, v) \in E$ the nodes u and v have different colours. From now on, whenever we speak of a *colouring* of a graph we mean a colouring with this property. The minimum number of colours needed to colour a given graph is called the *chromatic number* of the graph. Two main questions regarding graph colouring are: (i) Given a graph $\mathcal{G}$ from a certain class of graphs, how quickly can we compute its chromatic number? (ii) What is the chromatic number of a given graph class, that is, the smallest number of colours such that any graph from the class can be coloured with that many colours?

We are interested in these questions for graphs induced by geometric objects in the plane and, in particular, by contact graphs. Let $\mathcal{S} = \{P_1, \ldots, P_n\}$ be a set of geometric objects in the plane. The *intersection graph* induced by $\mathcal{S}$ is the graph whose nodes correspond to the objects in $\mathcal{S}$ and where there is an edge (P_i, P_j) if and only if P_i and P_j intersect. If the objects in $\mathcal{S}$ are closed and have disjoint interiors, then the intersection graph is called a *contact graph.* It has been shown that the class of contact graphs of discs is the same as the class of planar graphs: any contact graph of discs is planar and any planar graph can be drawn as a contact graph of discs [6]. By the Four-Colour Theorem [1] this implies that any contact graph of discs is 4-colourable. More generally, contact graphs of compact objects with smooth boundaries are planar, and so they are 4-colourable.

We are interested in colouring contact graphs of squares and rectilinear polygons. (Unless explicitly stated otherwise, whenever we speak of squares or rectilinear polygons we mean axis-parallel squares and axis-parallel rectilinear polygons.) Contact graphs of squares are different from contact graphs of smooth objects, because four (interior-disjoint) squares can all meet in a common point. Thus the obvious embedding of such a contact graph—where we put a node at the center of each square and we connect the centers of two touching squares by a two-link path through a touching point—is not necessarily plane.

Eppstein *et al.* [4] studied colourings of contact graphs of squares for the special cases where the squares form a quadtree subdivision, that is, the set $\mathcal{S}$ of squares is obtained by recursively subdividing an initial square in four equal-sized quadrants. They proved that any such contact graph is 6-colourable and they gave an example of a quadtree subdivision that requires five colours. (They also considered the variant where two squares that only touch in a single vertex are not considered neighbours.)

Our results. We start by studying the computational complexity of colouring contact graphs. We show that already for a set of unit squares, it is NP-complete to decide if the contact graph is 3-colourable.

Next we study the chromatic number of various classes of contact graphs. Recall that the obvious embedding of the contact graph of squares need not be plane. We first prove contact graphs of unit squares can have a K_m as a minor for an arbitrarily large m and, hence, need not be planar. Nevertheless, contact graphs of unit squares are 4-colourable and finding a 4-colouring is quite easy, so our NP-completeness result on 3-colouring completely characterizes the computational complexity of colouring unit squares. Contact graphs of arbitrarily-sized squares are not always 4-colourable—the quadtree example of Eppstein *et al.* [4] requiring five colours shows this. We prove that the chromatic number of the class of contact graphs of arbitrarily-sized squares is at most 6. In fact, we prove that any contact graph of a set of rectilinear polygons is 6-colourable. (Even more generally, contact graphs of polygons whose interior angles are strictly greater than $2\pi/5$ are 6-colourable.) Thus we obtain the same bound of Eppstein *et al.*, but for a much larger class of objects. Moreover, for this class the bound is tight. To prove our result, we charaterize contact graphs of rectilinear polygons as a certain subset of 1-planar graphs, which are known to be 6-colourable [2].

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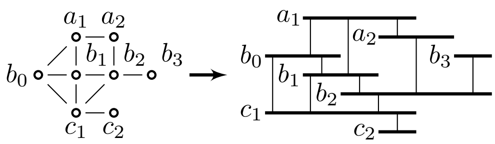
2 NP-Completeness of 3-Colourability

In this section we establish the hardness of 3-Colourability on contact graphs of unit squares.

Theorem 1 3-Colourability *on contact graphs of unit squares is* NP*-complete.*

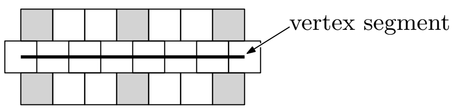
Proof. 3-Colourability on contact graphs is obviously in NP. To prove that the problem is NP-hard we use a reduction from the NP-complete problem of 3-colouring planar graphs of degree at most 4 [5].

Let $\mathcal{G} = (V, E)$ be any planar graph on n vertices with degree at most 4. Rosenstiehl and Tarjan [7] showed that we can compute in polynomial time a *visibility representation* of $\mathcal{G}$, in which every vertex $u \in V$ is represented by a horizontal *vertex segment* s_u and every edge $(u, v) \in E$ is represented by a vertical *edge segment* that connects s_u and s_v and does not intersect any other vertex segment.



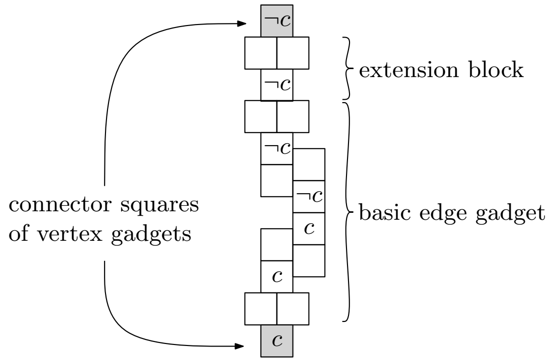
The construction can be done so that (i) all y-coordinates of the vertex segments are multiples of 10, and (ii) all x-coordinates of the edge segments are multiples of 3 and all x-coordinates of the left and right endpoints of the vertex segments are of the form $3i - \frac{1}{2}$ and $3j + \frac{1}{2}$, respectively, for some integers $i < j$.

The vertex gadget that replaces a vertex segment is as follows; the example shows the gadget for a segment of length 7.



The grey squares in the construction are called *connector squares.* In order to 3-colour a vertex gadget, all connector squares must receive the same colour. This colour represents the colour of the corresponding vertex in $\mathcal{G}$. Note that each edge segment passes through the center of a connector square on both vertex gadgets it connects.

The edge gadget that replaces an edge segment consists of a *basic edge gadget* plus zero or more *extension blocks.* Note that we can generate edge gadgets of vertical length $7 + 2j$ for any integer $j \geqslant 0$, by using j extension blocks. This suffices because the y-coordinates of the vertex segments are multiples of 10, and so the distance in between any two connector squares we need to connect by an edge is of the form $10k - 3$, for some integer $k \geqslant 1$. Our edge



gadget forces the connector squares of the two vertex gadgets it connects to have different colours. It is easily checked that this implies that the contact graph of the generated set of squares is 3-colourable if and only if the original graph $\mathcal{G}$ is. Moreover, the entire construction can be done in polynomial time. ☐

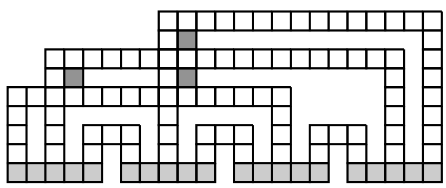
Using a similar proof we can show that 3-Colourability is NP-complete for contact graphs of discs, or of any other fixed convex and compact shape. Note that for discs (or other smooth shapes) this settles the complexity of the problem completely: contact graphs of smooth convex shapes are planar and so they are 4-colourable, and checking for 2-colourability is easy.

3 Unit Squares

If we draw the contact graph of a set of unit squares by putting vertices at the centers of the squares and drawing edges as straight segments, then the resulting drawing obviously need not be plane. The following theorem shows a stronger result, namely that contact graphs of unit squares are not planar and that in fact they can have a K_m-minor for arbitrarily large m.

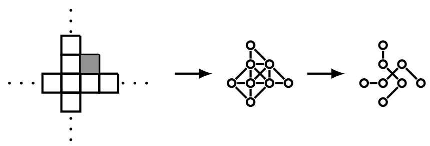
Theorem 2 *For any $m \geqslant 1$, there are contact graphs of unit squares with a K_m-minor.*

Proof. The squares we will generate to obtain a contact graph with a K_m as minor will all have integer coordinates. The following picture shows the construction for $m = 4$.



Next we explain the various components in the construction. Consider K_m. We call the nodes of the K_m *super nodes* and the edges *super edges.* For each super node u we put a block of $2m - 3$ unit squares whose lower edges all lie on the same horizontal line. The distance between two adjacent blocks is one unit. In the figure above, the blocks are the four light grey rectangles.

For each super edge (u, v) we create a path of squares as follows. We put two vertical columns of an even number of squares—one on top of the block created for u and one on top of the block created for v—which have the same height, and we connect the topmost squares of these columns by a row of squares. We can do this such that we do not create any adjacencies between squares from different paths, except where a column of one path crosses the row of another path. Note that in this case the two paths actually share a square. Where this happens we add one more square to the top-right of the shared square—see the three dark grey squares in the picture above. These extra square allow us to obtain a minor in which all super edges are represented by disjoint paths, as the next figure shows.



By contracting the (nodes corresponding to the) square in each block to a super node and contracting the paths connecting pairs of nodes into super edges we can now obtain our K_m as a minor. Note that the construction can be done with $O(m^4)$ squares (and we can show that at least $\Omega(m^4)$ are needed). ☐

Despite the fact that contact graphs of unit squares are not planar, they are 4-colourable.

Theorem 3 *Any contact graph of set of unit squares is 4-colourable, and this number is tight in the worst case.*

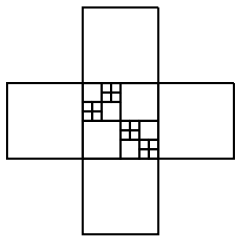
Proof. The lower bound construction is easy—just take four squares touching in a common point. For the upper bound, we divide the plane into horizontal strips of the form $(-\infty, +\infty) \times [i, i + 1)$ and assign each square to the strip containing its bottom edge. The squares assigned to a single strip can be coloured with only two colours, and by using the colour pair 1,2 for the strips with even i and 3,4 for the strips with odd i we obtain a 4-colouring. ☐

4 Arbitrarily-Sized Squares

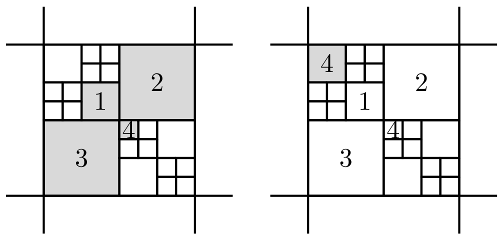
We now turn our attention to arbitrarily-sized squares.

Theorem 4 *Any contact graph of a set of squares is 6-colourable, and there are contact graphs of squares that need at least five colours.*

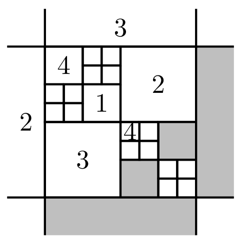
Proof. The upper bound follows from the result in the next section, where we show that even contact graphs of rectilinear polygons are 6-colourable. It remains to give an example of a set of squares that induces a contact graph that needs five colours. Eppstein *et al.* [4] already gave such an example (where the squares form a quadtree subdivision). For completeness we provide a different (and slightly smaller) example. We claim that the following graph (which is also the subgraph of a quadtree) needs at least 5 colours.



Suppose for a contradiction that the graph is 4-colourable. Then, without loss of generality, we can colour the four squares of the middle clique (consisting of four squares of different sizes) as depicted in the following picture (left). We claim that then the top left inner square has colour 4.



Indeed, if it has colour 2, none of the four squares on its right could use colour 2 and so one of these squares would need a fifth colour. Similarly, if it has colour 3, none of the four squares below it could use colour 3 and of those squares would need a fifth colour. Hence, it has to use colour 4 since it touches a square coloured with 1. Using similar arguments and simple deduction, we arrive to the following partial colouring:



Now we observe that the four gray squares form a cycle that surrounds a 4-clique. Moreover, we can easily deduce that none of the squares in the cycle can be coloured 2 or 3. Hence they have to use colour 1 and 4. But then the surrounded clique cannot use 1 or 4, a contradiction. We conclude that the graph is not 4-colourable. ☐

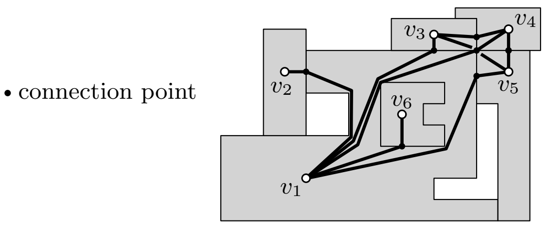
5 Rectilinear Polygons

We now turn our attention to contact squares of rectilinear polygons, where we allow the polygons to have holes. We will prove that such contact graphs are 6-colourable by showing that they are *1-planar graphs* [3], that is, graphs that can be drawn in the plane such that each edge has at most one crossing (that is, it crosses at most one other edge and this crossing then consists of a single point).

The following theorem establishes the exact relation between contact graphs of rectilinear polygons and 1-planar graphs. (We recently learned that a similar result, on the relation between 1-planar graphs and so-called 4-map graphs was already known [3]. Our proof concerns rectilinear maps and is more direct.)

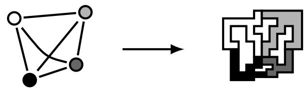
Theorem 5 *The class of contact graphs of rectilinear polygons is exactly the class of 1-plane graphs in which every pair of crossing edges is part of a K_4.*

Proof. Let $\mathcal{S} := \{P_1, \ldots, P_n\}$ be a set of interior-disjoint rectilinear polygons. To prove that the contact graph of $\mathcal{S}$ is 1-planar, we proceed as follows. First we add a point v_i in the interior of every polygon P_i, which is the embedding of the node corresponding to P_i. Next, for each pair of touching polygons P_i, P_j we pick a *connection point* $q_{ij} \in \partial P_i \cap \partial P_j$. If $\partial P_i \cap \partial P_j$ has non-zero length, we pick q_{ij} in the relative interior of $\partial P_i \cap \partial P_j$. We then embed the edge (v_i, v_j) by the union of two paths from q_{ij}: a path $\pi(q_{ij}, v_i) \subset P_i$ to v_i and a path $\pi(q_{ij}, v_j) \subset P_j$ to v_j. We do this in such a way that, for each P_i, the paths from the connection points on ∂P_i to v_i are pairwise disjoint (except at their shared endpoint v_i). This



can always be done, for example by taking a shortest-path tree rooted at v_i whose leaves are the connection points on ∂P_i. Thus an edge (v_i, v_j) can only intersect an edge (v_k, v_ℓ) when $q_{ij} = q_{k\ell}$. Since any point can be a connection point for at most two pairs of polygons, this means that in our embedding every edge intersects at most one other edge. Moreover, since all four polygons meet on the crossing point, they are part of a 4-clique.

Next we show that every 1-planar graph $\mathcal{G} = (V, E)$ with every pair of crossing edges forming a K_4 is the contact graph of a set of rectilinear polygons. Such a set can be obtained from a "pixelised" image of a 1-planar drawing of $\mathcal{G}$.

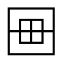


Each polygon is obtained by the vertex it represents and half of each of its edges, as shown in the picture above. If the edge is not crossing any other, we can decide arbitrarily where to divide it into the two polygons. If it crosses another edge, we cut it at the crossing point, as depicted above.

We can actually obtain a suitable set of polygons whose total number of vertices is linear in $O(|V|)$, but the proof is more complex. This bound is tight since we can construct an instance where one of the polygons has linear complexity: since for each crossing we need a corner, it suffices to have a vertex with a linear number of edges crossing other edges. $\square$

Note that this proof works for non-rectilinear polygons as long as no five of them touch on a single point, which is always satisfied when the interior angles are strictly bigger than $2\pi/5$.

Since 1-planar graphs are 6-colourable and the figure below shows a rectilinear representation of K_6, Theorem 5 immediately implies the following.



Corollary 6 *Any contact graph of a set of rectilinear polygons is 6-colourable, and this number is tight in the worst case.*

Acknowledgment

We thank one reviewer for pointing out reference [3].

References

[1] K. Appel and W. Haken. Every map is four colourable. *Bulletin of the American Mathematical Society*, 82:711–712, 1976.

[2] O. V. Borodin. A new proof of the 6 color theorem. *Journal of Graph Theory*, 19(4):507–521, 1995.

[3] F. J. Brandenburg. On 4-map graphs and 1-planar graphs and their recognition problem. *CoRR*, abs/1509.03447, 2015.

[4] D. Eppstein, M. W. Bern, and B. L. Hutchings. Algorithms for coloring quadtrees. *Algorithmica*, 32(1):87–94, 2002.

[5] M. Garey, D. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1(3):237 – 267, 1976.

[6] P. Koebe. Kontaktprobleme der konformen abbildung. *Ber. Verh. Sächs. Akademier der Wissenschaften Leipzig, Math.-Phys. Klasse 88*, pages 141–164, 1936.

[7] P. Rosenstiehl and R. E. Tarjan. Rectilinear planar layouts and bipolar orientations of planar graphs. *Discrete & Computational Geometry*, 1:343–353, 1986.