Detecting affine equivalences of planar rational curves

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Abstract

We derive a system of polynomial equations to decide whether two rational parametric curves in the plane are related by an affine transformation and to detect all such affine equivalences. In order to do so, we use homogenization in both the parameter domain and the Euclidean plane. Furthermore, employing barycentric coordinates leads to a simple method for detecting affine equivalences, as these coordinates are invariant under affine transformations. In addition we interpret the result by relating the monomial coefficients to Bézier control points. Finally we provide numerical examples.

1 Introduction

Detecting symmetries is an essential problem in Pattern Recognition, Computer Graphics and Computer Vision. First approaches concentrated on point sets as input data. In 2004, Braß and Knauer [5] proposed to apply a point-based method to control polygons of Bézier curves and surfaces. For matching planar curve segments in B-spline form, a method based on affinely invariant moments has been described in [6]. Sánchez-Reyes [8] recently developed a method for symmetry detection of curves given in Bernstein-Bézier representation. Lebmeir and Richter-Gebert [7] investigated symmetries of algebraic curves given in implicit form. During the last two years, Alcázar et al. [1, 2, 3, 4] published a series of papers dealing with the problem of symmetry detection for parametric rational curves. They use the fact that the symmetry of a curve in proper parameterization can be related to a rational linear transformation in the parameter domain, see [9].

We consider properly parameterized rational curves and investigate the more general concept of affine equivalences. Symmetry detection can then be seen as a special case.

2 Detecting equivalences

Before presenting our method, we recall some geometric tools and clarify our notation.

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2.1 Preliminaries

We consider curves in the projectively closed Euclidean plane $\bar{E}^2$, whose points are given by homogeneous coordinate vectors $\mathbf{x} = (x_0, x_1, x_2)^T \in \mathbb{R}^3 \backslash \{(0,0,0)\}$. If there exists a $\mu \neq 0$, such that $\mathbf{x} = \mu\mathbf{y}$, $\mathbf{x}$ and $\mathbf{y}$ represent the same point in $\bar{E}^2$. We denote this by $\mathbf{x} \simeq \mathbf{y}$.

Three non-collinear base points $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$, none of which is a point at infinity, define a barycentric coordinate system, such that any finite point $\mathbf{x}$ possesses unique barycentric coordinates $\lambda_i(\mathbf{x})$, $i = 0, \ldots, 2$, with respect to the base points. More precisely, we have

$$\frac{1}{x_0}\mathbf{x} = \sum_{i=0}^{2} \lambda_i(\mathbf{x}) \frac{1}{v_{i,0}} \mathbf{v}_i.$$

The barycentric coordinates can be computed using homogeneous coordinates

$$\begin{aligned} \lambda_0(\mathbf{x}; \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) &= \Lambda(\mathbf{x}; \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2), \\ \lambda_1(\mathbf{x}; \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) &= \Lambda(\mathbf{x}; \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0), \\ \lambda_2(\mathbf{x}; \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) &= \Lambda(\mathbf{x}; \mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1) \end{aligned}$$

where

$$\Lambda(\mathbf{x}; \mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{a_0 \det(\mathbf{x}, \mathbf{b}, \mathbf{c})}{x_0 \det(\mathbf{a}, \mathbf{b}, \mathbf{c})}. \tag{1}$$

Throughout the paper we consider two parametric rational curves $\mathcal{C}$ (and $\mathcal{C}'$, respectively) $\subset \bar{E}^2$, which are considered as point sets. Both curves are given by proper parameterizations[1]

$$\begin{aligned} \mathbf{p} : P^1(\mathbb{R}) \to \mathcal{C} \subset \bar{E}^2, \\ \mathbf{t} \mapsto \mathbf{p}(\mathbf{t}) = (p_0(t_0, t_1), p_1(t_0, t_1), p_2(t_0, t_1)). \end{aligned}$$

The parameter $\mathbf{t} = (t_0, t_1)$ is a point on the projective line $P^1(\mathbb{R})$.

The homogeneous coordinates of the curves are homogeneous polynomials of degree n,

$$p_j(\mathbf{t}) = \sum_{i=0}^{n} c_{j,i} t_0^{n-i} t_1^i$$

with coefficient vectors

$$\mathbf{c}_i = (c_{0,i}, c_{1,i}, c_{2,i})^T.$$

Polynomials of degree n given in standard (i.e., non-homogeneous) form are homogenized by replacing t^i with $t_0^{n-i} t_1^i$.

[1] If improper parameterizations are given, one may obtain proper ones by applying a suitable reparameterization, see [9].

Furthermore we assume that both curves are in reduced form, i.e.

$$\gcd(p_0(\mathbf{t}), p_1(\mathbf{t}), p_2(\mathbf{t})) = \gcd(p'_0(\mathbf{t}), p'_1(\mathbf{t}), p'_2(\mathbf{t})) = 1$$

and of common degree $n \geq 2$. Affine transformation do not change the degree of a curve.

In particular, this implies that both curves possess the same degree

$$\max(\deg_{t_i}(p_0(\mathbf{t})), \deg_{t_i}(p_1(\mathbf{t})), \deg_{t_i}(p_2(\mathbf{t}))) = n,$$
$$\max(\deg_{t_i}(p'_0(\mathbf{t})), \deg_{t_i}(p'_1(\mathbf{t})), \deg_{t_i}(p'_2(\mathbf{t}))) = n,$$

with respect to t_i, $i = 0, 1$. Note that $n \geq 2$ exludes lines, since we consider proper parameterizations only.

Recall that using homogeneous coordinates allows to represent any affine transformation by a matrix multiplication

$$\mathbf{x} \mapsto M\mathbf{x}, \quad M = \begin{pmatrix} 1 & 0 \\ \vec{\mathbf{b}} & A \end{pmatrix},$$

where A is a 2×2 matrix and $\vec{\mathbf{b}} \in \mathbb{R}^2$. Any regular affine transformation is represented by a non-singular matrix M. The class of affine transformations includes translations, rotations, uniform and non-uniform scalings, reflections and shears.

Definition 1 Two curves $\mathcal{C}$ and $\mathcal{C}'$ are said to be *affinely equivalent* if there exists a regular affine transformation matrix M such that $\mathcal{C}' = M\mathcal{C}$. Furthermore, $\mathcal{C}$ is said to possess an *affine symmetry* if there exists a regular affine transformation matrix M, different from the identity, such that $\mathcal{C} = M\mathcal{C}$.

Due to the group structure of regular affine mappings, affine equivalences define an equivalence relation. If the matrix A is orthogonal, i.e. $A^T A = I$, then affinely equivalent curves are said to be *congruent* and an affine symmetry is simply called a *symmetry*. If A is a multiple of an orthogonal matrix, $A^T A = \lambda I$ with $\lambda \in \mathbb{R}$, then the affinely equivalent curves are said to be *similar*.

2.2 Coefficient-based detection

Lemma 1 *Two rational parameterizations* $\mathbf{p}(\mathbf{t})$ *and* $\mathbf{p}'(\mathbf{t})$ *are equivalent, i.e.* $\mathbf{p}(\mathbf{t}) \simeq \mathbf{p}'(\mathbf{t})$ *holds for all* $\mathbf{t} \in P^1(\mathbb{R})$*, if and only if there exists a non-zero constant* μ *such that* $\mathbf{c}_i = \mu\mathbf{c}'_i$*,* $i = 0, \ldots, n$*.*

Proof. The equivalence of the two curves implies that there exists a rational function

$$\mu(\mathbf{t}) = \frac{\mu_1(\mathbf{t})}{\mu_0(\mathbf{t})} = \frac{p'_0(\mathbf{t})}{p_0(\mathbf{t})} = \frac{p'_1(\mathbf{t})}{p_1(\mathbf{t})} = \frac{p'_2(\mathbf{t})}{p_2(\mathbf{t})}$$

where μ_0 are μ_1 are relatively prime polynomials, such that $\mathbf{p}(\mathbf{t}) = \mu(\mathbf{t})\mathbf{p}'(\mathbf{t})$. Consequently, the two rational curves satisfy

$$\mu_0(\mathbf{t})\mathbf{p}(\mathbf{t}) = \mu_1(\mathbf{t})\mathbf{p}'(\mathbf{t}).$$

This function is indeed a constant since

$$\mu_0 | \underbrace{\gcd(p'_0, p'_1, p'_2)}_{=1} \quad \text{and} \quad \mu_1 | \underbrace{\gcd(p_0, p_1, p_2)}_{=1}.$$

□

Recall that any two proper parameterizations of a rational curve are related by a linear rational reparameterization, which is simply a regular projective transformation of the real projective line

$$\mathbf{r}(\mathbf{t}) = \underbrace{\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}}_{=\alpha} \mathbf{t} = \begin{pmatrix} \alpha_{00}t_0 + \alpha_{01}t_1 \\ \alpha_{10}t_0 + \alpha_{11}t_1 \end{pmatrix}$$

described by a regular matrix α. We investigate the transformation of the coefficients which is caused by such a reparameterization.

Lemma 2 *The reparameterized curve* $\hat{\mathbf{p}} = \mathbf{p} \circ \mathbf{r}$*,*

$$\mathbf{p}(\mathbf{r}(\mathbf{t})) = \hat{\mathbf{p}}(\mathbf{t}) = \sum_{j=0}^{n} \hat{\mathbf{c}}_j t_0^{n-j} t_1^j$$

has the coefficients

$$\hat{\mathbf{c}}_j(\alpha) = \sum_{i=0}^{n} \mathbf{c}_i \sum_{\ell=0}^{j} \binom{n-i}{\ell} \binom{i}{j-\ell} \alpha_{00}^{n-i-\ell} \alpha_{01}^{\ell} \alpha_{10}^{i-j+\ell} \alpha_{11}^{j-\ell}$$

for $j = 0, \ldots, n$*.*

Proof. This result is confirmed by a simple computation and by comparing the coefficients. □

We identify affine equivalences by analyzing whether the coefficients are related by an affine transformation.

Proposition 3 *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be rational planar curves with parameterizations* $\mathbf{p}(t)$ *and* $\mathbf{p}'(t)$ *satisfying our assumptions. The two curves are affinely equivalent if and only if there exists a constant* μ*, an affine transformation matrix* M *and a regular projective transformation* α*, such that the control points of both curves satisfy*

$$M\mathbf{c}'_j = \mu\,\hat{\mathbf{c}}_j(\alpha), \quad j = 0, \ldots, n. \tag{2}$$

Proof. On the one hand, the conditions (2) imply that the two curves are affinely equivalent. On the other hand, we consider two affinely invariant curves $\mathcal{C}'$ and $\mathcal{C}$. There exists an affine transformation M such that

$$M\mathcal{C}' = \mathcal{C}.$$

We define $\mathbf{z}(\mathbf{t}) = M\mathbf{p}'(\mathbf{t})$. Consequently $\mathbf{z}(\mathbf{t})$ and $\mathbf{p}(\mathbf{t})$ are two proper parameterizations of the same curve $\mathcal{C}$. According to Lemma 4.17 of [9] there is a linear rational reparameterization $\mathbf{r}(\mathbf{t})$ – and hence an associated projective transformation α – such that

$$\mathbf{z}(\mathbf{t}) \simeq \mathbf{p}(\mathbf{r}(\mathbf{t})).$$

Thus we obtain that

$$\sum_{i=0}^{n} M\mathbf{c}'_i t_0^{n-i} t_1^i = M\mathbf{p}'(\mathbf{t}) = \mathbf{z}(\mathbf{t}) \simeq \mathbf{p}(\mathbf{r}(\mathbf{t})) = \hat{\mathbf{p}}(\mathbf{t}) = \sum_{i=0}^{n} \hat{\mathbf{c}}_i(\alpha) t_0^{n-i} t_1^i.$$

Using Lemma 1 confirms (2). □

2.3 Barycentric coordinates

The existence of the affine transformation matrix M can be characterized with the help of barycentric coordinates.

Corollary 4 *Let $\mathcal{C}$ and $\mathcal{C}'$ be two rational planar curves as in Proposition 3. We assume that*

(i) all points $\mathbf{c}'_i$ are finite points ($c'_{i,0} \neq 0$) and

(ii) the first three points $\mathbf{c}'_0$, $\mathbf{c}'_1$ and $\mathbf{c}'_2$ are non-collinear.

The two curves $\mathcal{C}$ and $\mathcal{C}'$ are affinely equivalent if and only if there exist a regular projective transformation α and a constant μ such that the equations

$$c'_{j,0} = \mu\, \hat{c}_{j,0}(\alpha), \quad j = 0, \dots, n \tag{3}$$

and

$$\lambda_i(\mathbf{c}'_j; \mathbf{c}'_0, \mathbf{c}'_1, \mathbf{c}'_2) = \lambda_i(\hat{\mathbf{c}}_j(\alpha); \hat{\mathbf{c}}_0(\alpha), \hat{\mathbf{c}}_1(\alpha), \hat{\mathbf{c}}_2(\alpha)), \quad i = 0, \dots, 2, \quad j = 3, \dots, n. \tag{4}$$

are satisfied.

For any solution of the system (3) and (4), we obtain the corresponding affine transformation by solving the linear system of equations in six unknowns

$$M\mathbf{c}'_i = \mu\hat{\mathbf{c}}_i \qquad \text{for } i = 0, \dots, 2.$$

In order to find Euclidean congruences and Euclidean symmetries (resp. similarities) we have to check in a postprocessing step whether the submatrix A is orthogonal (resp. a multiple of an orthogonal matrix).

Clearly, it is also possible to consider other triplets of points in (ii) and (4).

2.4 The case of Bézier control points

Rational Bézier curves of degree n

$$\mathbf{p}(u) = \sum_{i=0}^{n} B_i^n(u)\mathbf{b}_i$$

generally possess the properties (proper parameterization, reduced form, common denominator) which are assumed by our method. These curves can be homogenized by simply replacing the Bernstein polynomials $B_i^n(u)$ by $\binom{n}{i} t_0^{n-i} t_1^i$. This is equivalent to the standard homogenization $u = \frac{u_1}{u_0}$ and a multiplication by u_0^n, followed by the projective transformation

$$\begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_0 - u_1 \\ u_1 \end{pmatrix}.$$

of the parameter domain. That means that the control points $\mathbf{b}_i$ of the Bézier curves are related to the monomial coefficients after this transformation via

$$\mathbf{c}_i = \binom{n}{i} \mathbf{b}_i.$$

3 Implementation and Examples

If the two conditions (i) or (ii) are not all satisfied then we apply an arbitrary projective transformation α' to the parameterization $\mathbf{p}'$ of the second curve. Note that the first condition is always violated for polynomial curves, hence a reparameterization is needed in this situation. The reparameterized curve $\hat{\mathbf{p}}' = \mathbf{p}' \circ \mathbf{r}'$, where $\mathbf{r}'$ is defined by α', satisfies all conditions in general and its control points are obtained from Lemma 2.

From equations (3) and (4) we obtain a system of polynomial equations in α and μ by using (1). Without loss of generality we may apply the normalization $|\mu| = 1$ and arrive at the equations

$$c'_{j,0} = \pm\, \hat{c}_{j,0}(\alpha), \quad j = 0, \dots, n$$

and

$$\lambda_i(\mathbf{c}'_j; \mathbf{c}'_0, \mathbf{c}'_1, \mathbf{c}'_2) = \lambda_i(\hat{\mathbf{c}}_j(\alpha); \hat{\mathbf{c}}_0(\alpha), \hat{\mathbf{c}}_1(\alpha), \hat{\mathbf{c}}_2(\alpha)), \quad i = 0, \dots, 2, \quad j = 3, \dots, n.$$

These form a system in four unknowns α consisting of $3n - 3$ equations, since we may omit the equations obtained for $i = 2$ as the barycentric coordinates sum to 1.

We performed the computations using Mathematica Version 10, where we used the built-in functions Reduce[] and NSolve[] to solve the system by symbolic and numeric computations, respectively. For every example we considered affine symmetries, and affine equivalences with ($\mathbf{p}'$) and without ($\mathbf{p}''$) reparameterization. More precisely, we considered a master curve and two curves derived from it by applying affine transformations and parameter transformations.

The first example is the lemniscate (Fig. 1), which is a degree 4 curve given by

$$t \mapsto \begin{pmatrix} 1 + 4t + 12t^2 + 16t^3 + 8t^4 \\ 1 + 4t + 6t^2 + 4t^3 \\ 2t + 6t^2 + 4t^3 \end{pmatrix}.$$

example	deg.	# of equiv.	$\mathbf{p(t)}$ NSolve	$\mathbf{p(t)}$ Reduce	$\mathbf{p(t)}$ and $\mathbf{p'(t)}$ NSolve	$\mathbf{p(t)}$ and $\mathbf{p'(t)}$ Reduce	$\mathbf{p(t)}$ and $\mathbf{p''(t)}$ NSolve	$\mathbf{p(t)}$ and $\mathbf{p''(t)}$ Reduce
lemniscate	4	4	0.33	0.25	0.33	0.23	0.41	0.45
epitrochoid	4	2	0.14	0.3	0.13	0.3	0.14	0.34
4-leaf rose	6	8	2.65	2.53	2.68	1.97	2.34	2.45
offset of a cardioid	8	2	1.03	20.04	1.03	19.91	1.06	31.76

Table 1: Computation times (times in seconds)

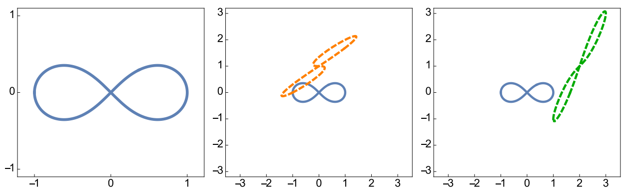

Figure 1: Several lemniscate like curves

As a second example we investigate the epitrochoid (see Fig. 2), given by

$$t \mapsto \begin{pmatrix} 7 + 28t + 56t^2 + 56t^3 + 28t^4 \\ 1 + 4t + 24t^2 + 40t^3 + 12t^4 \\ 4t + 12t^2 - 8t^3 - 16t^4 \end{pmatrix}.$$

Again we applied reparameterizations and affine mappings (not shown), similar to the previous example.

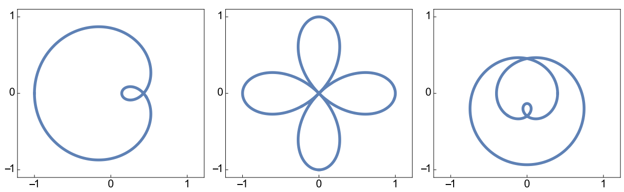

Figure 2: Epitrochoid, 4-leaf rose and offset of a cardioid.

The third example, the 4-leaf rose (see again Fig. 2), is given by the parameterization

$$t \mapsto \begin{pmatrix} 11 + 6t + 18t^2 + 32t^3 + 36t^4 + 24t^5 + 8t^6 \\ 2t + 10t^2 + 8t^3 - 16t^4 - 24t^5 - 8t^6 \\ 1 + 6t + 8t^2 - 8t^3 - 20t^4 - 8t^5 \end{pmatrix}$$

of degree 6. Finally we apply our algorithm to the offset of a cardioid (see again Fig. 2),

$$t \mapsto \begin{pmatrix} 15(6561 + 2916t^2 + 486t^4 + 36t^6 + t^8) \\ -39366 + 61236t^2 - 31104t^3 + 3456t^5 - 756t^6 + 6t^8 \\ -18t(4374 - 1296t - 1134t^2 + 864t^3 - 126t^4 - 16t^5 + 6t^6) \end{pmatrix}$$

which is a rational curve of degree 8.

Table 1 presents the computation times (on a standard PC) for solving the system (3) and (4) in these examples.

4 Conclusion

We presented a method to detect affine equivalences of planar rational curves. For moderate degrees of the input curve, the corresponding polynomial system can be solved within seconds using standard computer algebra tools. To the best of our knowledge, this is the first work on detecting affine equivalences and affine symmetries of rational curves, and it also encompasses the computation of symmetries or similarities, which was studied by several authors [1, 2, 3, 4, 5, 7, 8], as special cases. Future work will be devoted to the generalization to higher dimensions and to the detection of approximate affine equivalences via numerical methods.

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