Bottleneck Matchings and Hamiltonian Cycles in Higher-Order Gabriel Graphs*

Ahmad Biniaz† Anil Maheshwari† Michiel Smid†

Abstract

Given a set P of n points in the plane, the order-k Gabriel graph on P, denoted by k-GG, has an edge between two points p and q if and only if the closed disk with diameter pq contains at most k points of P, excluding p and q. It is known that 10-GG contains a Euclidean bottleneck matching of P, while 8-GG may not contain such a matching. We answer the following question in the affirmative: does 9-GG contain any Euclidean bottleneck matching of P?

It is also known that 10-GG contains a Euclidean bottleneck Hamiltonian cycle of P, while 5-GG may not contain such a cycle. We improve the lower bound and show that 7-GG may not contain any Euclidean bottleneck Hamiltonian cycle of P.

1 Introduction

Let P be a set of n points in the plane. For any two points $p, q \in P$, let $D[p, q]$ denote the closed disk that has the line segment $\overline{pq}$ as diameter. Let $|pq|$ be the Euclidean distance between p and q. The *Gabriel graph* on P, denoted by $GG(P)$, is a geometric graph that has an edge between two points p and q if and only if $D[p, q]$ does not contain any point of $P \setminus \{p, q\}$. Gabriel graphs were introduced by Gabriel and Sokal [6] and can be computed in $O(n \log n)$ time [8]. Every Gabriel graph has at most $3n - 8$ edges, for $n \geq 5$, and this bound is tight [8].

The *order-k Gabriel graph* on P, denoted by k-GG, is the geometric graph that has an edge between two points p and q if and only if $D[p, q]$ contains at most k points of $P \setminus \{p, q\}$. Thus, the Gabriel graph, $GG(P)$, corresponds to 0-GG. Su and Chang [9] showed that k-GG can be constructed in $O(k^2 n \log n)$ time and contains $O(k(n - k))$ edges. For two points $p, q \in P$, the *lune* of p and q, denoted by $L(p, q)$, is defined as the intersection of the two open disks of radius $|pq|$ centered at p and q. The *order-k Relative Neighborhood Graph* on P, denoted by k-RNG, is the geometric graph that has an edge (p, q) if and only if $L(p, q)$ contains at most k points of P. Note that k-RNG on P is a subgraph of k-GG on P.

A *matching* in a graph G is a set of edges without common vertices. A *perfect matching* is a matching that matches all the vertices of G. A *Hamiltonian cycle* in G is a cycle that visits each vertex of G exactly once. In the case when G is an edge-weighted graph, a *bottleneck matching* is defined to be a perfect matching in G, in which the weight of the maximum-weight edge is minimized. Moreover, a *bottleneck Hamiltonian cycle* is a Hamiltonian cycle in G, in which the weight of the maximum-weight edge is minimized. For a point set P, a *Euclidean bottleneck matching* is a perfect matching in the complete graph with vertex set P that minimizes the longest edge; the weight of an edge is defined to be the Euclidean distance between its two endpoints. Similarly, a *Euclidean bottleneck Hamiltonian cycle* is a Hamiltonian cycle that minimizes the longest edge.

Chang et al. [4] proved that a Euclidean bottleneck matching of P is contained in 16-RNG.[1] This implies that 16-GG contains a Euclidean bottleneck matching. In [2] the authors improved the bound for the latter graphs by showing that 10-GG contains a Euclidean bottleneck matching. They also show that 8-GG may not have any Euclidean bottleneck matching. They asked if 9-GG contains any Euclidean bottleneck matching. In Section 2, we answer this question in the affirmative.

Theorem 1 *For every point set P, 9-GG contains a Euclidean bottleneck matching of P.*

Chang et al. [3] proved that a Euclidean bottleneck Hamiltonian cycle of P is contained in 19-RNG, which implies that 19-GG contains a Euclidean bottleneck Hamiltonian cycle. Abellanas et al. [1] improved the bound by showing that 15-GG contains a Euclidean bottleneck Hamiltonian cycle. Kaiser et al. [7] improved the bound further by showing that 10-GG contains a Euclidean bottleneck Hamiltonian cycle. They also provide an example which shows that 5-GG may not contain any Euclidean bottleneck Hamiltonian cycle. In Section 3, we improve the lower bound to 7 and prove the following proposition.

Proposition 1 *There exist point sets P such that 7-GG does not contain any Euclidean bottleneck Hamiltonian cycle of P.*

*Research supported by NSERC.

†Carleton University, Ottawa, Canada.

[1] They defined k-RNG to have an edge (p, q) if and only if $L(p, q)$ contains at most $k - 1$ points of P.

Therefore, it remains open to decide whether or not 8-*GG* or 9-*GG* contains a Euclidean bottleneck Hamiltonian cycle.

2 Proof of Theorem 1

In this section we prove Theorem 1. The proofs for Lemmas 2 and 3 are similar to the proofs in [4] which are adjusted for Gabriel graphs. The proof of Lemma 4 is based on a similar technique that is used in [7] for the Hamiltonicity of Gabriel graphs.

Let $\mathcal{M}$ be the set of all perfect matchings of the complete graph with vertex set P. For a matching $M \in \mathcal{M}$ we define the *weight sequence* of M, $\mathrm{WS}(M)$, as the sequence containing the weights of the edges of M in non-increasing order. A matching M_1 is said to be less than a matching M_2 if $\mathrm{WS}(M_1)$ is lexicographically smaller than $\mathrm{WS}(M_2)$. We define a total order on the elements of $\mathcal{M}$ by their weight sequence. If two elements have exactly the same weight sequence, break ties arbitrarily to get a total order.

Let $M^* = \{(a_1, b_1), \dots, (a_{\frac{n}{2}}, b_{\frac{n}{2}})\}$ be a matching in $\mathcal{M}$ with minimum weight sequence. Observe that M^* is a Euclidean bottleneck matching for P. In order to prove Theorem 1, we will show that all edges of M^* are in 9-*GG*. Consider any edge (a, b) in M^*. If $D[a, b]$ contains no point of $P \setminus \{a, b\}$, then (a, b) is an edge of 9-*GG*. Suppose that $D[a, b]$ contains k points of $P \setminus \{a, b\}$. We are going to prove that $k \le 9$. Let $R = \{r_1, r_2, \dots, r_k\}$ be the set of points of $P \setminus \{a, b\}$ that are in $D[a, b]$. Let $S = \{s_1, s_2, \dots, s_k\}$ represent the points for which $(r_i, s_i) \in M^*$.

Without loss of generality, we assume that $D[a, b]$ has diameter 1 and is centered at the origin $o = (0, 0)$, and $a = (-0.5, 0)$ and $b = (0.5, 0)$. For any point p in the plane, let $\|p\|$ denote the distance of p from o. Note that $|ab| = 1$, and for any point $x \in D[a, b] \setminus \{a, b\}$ we have $\max\{|xa|, |xb|\} < 1$.

Lemma 2 *For each point $s_i \in S$, $\min\{|s_i a|, |s_i b|\} \ge 1$.*

Proof. The proof is by contradiction; suppose that $|s_i a| < 1$. Let M be the perfect matching obtained from M^* by deleting $\{(a, b), (r_i, s_i)\}$ and adding $\{(s_i, a), (r_i, b)\}$. The lengths of the two new edges are smaller than 1, and hence both (s_i, a) and (r_i, b) are shorter than (a, b). Thus, $\mathrm{WS}(M) <_{\mathrm{lex}} \mathrm{WS}(M^*)$, which contradicts the minimality of M^*. ☐

As a corollary of Lemma 2, R and S are disjoint.

Lemma 3 *For each pair of points $s_i, s_j \in S$, $|s_i s_j| \ge \max\{|r_i s_i|, |r_j s_j|, 1\}$.*

Proof. The proof is by contradiction; suppose that $|s_i s_j| < \max\{|r_i s_i|, |r_j s_j|, 1\}$. Let M be the perfect matching obtained from M^* by deleting $\{(a, b), (r_i, s_i), (r_j, s_j)\}$ and adding $\{(a, r_i), (b, r_j), (s_i, s_j)\}$. Note that $\max\{|ar_i|, |br_j|, |s_i s_j|\} < \max\{|r_i s_i|, |r_j s_j|, |ab|\}$. Thus, $\mathrm{WS}(M) <_{\mathrm{lex}} \mathrm{WS}(M^*)$, which contradicts the minimality of M^*. ☐

Let $C(x, r)$ (resp. $D(x, r)$) be the circle (resp. closed disk) of radius r that is centered at a point x in the plane. For $i \in \{1, \dots, k\}$, let s'_i be the intersection point between $C(o, 1.5)$ and the ray with origin at o passing through s_i. Let the point p_i be s_i, if $\|s_i\| < 1.5$, and s'_i, otherwise. See Figure 1. Let $S' = \{a, b, p_1, \dots, p_k\}$.

Observation 1 *Let s_j be a point in S, where $\|s_j\| \ge 1.5$. Then, the disk $D(s_j, \|s_j\| - 0.5)$ is contained in the disk $D(s_j, |s_j r_j|)$. Moreover, the disk $D(p_j, 1)$ is contained in the disk $D(s_j, \|s_j\| - 0.5)$. See Figure 1.*

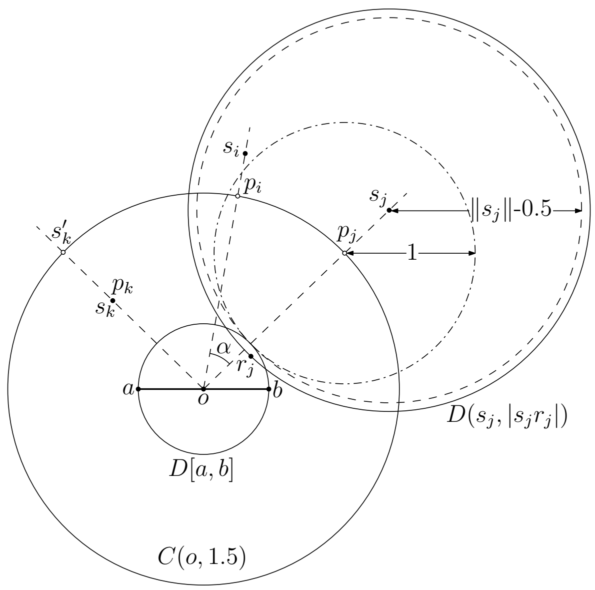

Figure 1: Proof of Lemma 4; $p_i = s'_i$, $p_j = s'_j$, and $p_k = s_k$.

Lemma 4 *The distance between any pair of points in S' is at least 1.*

Proof. Let x and y be two points in S'. We are going to prove that $|xy| \ge 1$. We distinguish between the following three cases.

- $\{x, y\} = \{a, b\}$. In this case the claim is trivial.
- $x \in \{a, b\}, y \in \{p_1, \dots, p_k\}$. If $\|y\| = 1.5$, then y is on $C(o, 1.5)$, and hence $|xy| \ge 1$. If $\|y\| < 1.5$, then y is a point in S. Therefore, by Lemma 2, $|xy| \ge 1$.
- $x, y \in \{p_1, \dots, p_k\}$. Without loss of generality assume $x = p_i$ and $y = p_j$, where $1 \le i < j \le k$. We differentiate between three cases:

Case (i): $\|p_i\| < 1.5$ and $\|p_j\| < 1.5$. In this case p_i and p_j are two points in S. Therefore, by Lemma 3, $|p_ip_j| \geq 1$.

Case (ii): $\|p_i\| < 1.5$ and $\|p_j\| = 1.5$. In this case p_i is a point in S. By Observation 1, the disk $D(p_j, 1)$ is contained in the disk $D(s_j, |s_jr_j|)$, and by Lemma 3, p_i is not in the interior of $D(s_j, |s_jr_j|)$. Therefore, p_i is not in the interior of $D(p_j, 1)$, which implies that $|p_ip_j| \geq 1$.

Case (iii): $\|p_i\| = 1.5$ and $\|p_j\| = 1.5$. In this case $\|s_i\| \geq 1.5$ and $\|s_j\| \geq 1.5$. Without loss of generality assume $\|s_i\| \leq \|s_j\|$. For the sake of contradiction assume that $|p_ip_j| < 1$; see Figure 1. Then, for the angle $\alpha = \angle s_ios_j$ we have $\sin(\alpha/2) < \frac{1}{3}$. Then, $\cos(\alpha) = 1 - 2\sin^2(\alpha/2) > \frac{7}{9}$. By the law of cosines in the triangle $\triangle s_ios_j$, we have

$$|s_is_j|^2 < \|s_i\|^2 + \|s_j\|^2 - \frac{14}{9}\|s_i\|\|s_j\|. \quad (1)$$

By Observation 1, the disk $D(s_j, \|s_j\| - 0.5)$ is contained in the disk $D(s_j, |s_jr_j|)$, and by Lemma 3, s_i is not in the interior of $D(s_j, |s_jr_j|)$. Therefore, s_i is not in the interior of $D(s_j, \|s_j\| - 0.5)$. Thus, $|s_is_j| \geq \|s_j\| - 0.5$. In combination with Inequality (1), this implies

$$\|s_j\|\left(\frac{14}{9}\|s_i\| - 1\right) < \|s_i\|^2 - \frac{1}{4}. \quad (2)$$

In combination with the assumption $\|s_i\| \leq \|s_j\|$, Inequality (2) implies

$$\frac{5}{9}\|s_i\|^2 - \|s_i\| + \frac{1}{4} < 0,$$

i.e.,

$$\frac{5}{9}\left(\|s_i\| - \frac{3}{10}\right)\left(\|s_i\| - \frac{3}{2}\right) < 0.$$

This is a contradiction, because, since $\|s_i\| \geq 1.5$, the left-hand side is non-negative. Thus $|p_ip_j| \geq 1$, which completes the proof of the lemma.

□

By Lemma 4, the points in S' have mutual distance at least 1. Moreover, the points in S' lie in $D(o, 1.5)$. Fodor [5] proved that the smallest circle which contains 12 points with mutual distances at least 1 has radius 1.5148. Therefore, S' contains at most 11 points. Since $a, b \in S'$, this implies that $k \leq 9$. Therefore, S, and consequently R, contains at most 9 points. Thus, (a, b) is an edge in 9-GG. This completes the proof of Theorem 1.

3 Proof of Proposition 1

In this section we prove Proposition 1. We show that for some point sets P, 7-GG does not contain any Euclidean bottleneck Hamiltonian cycle of P.

Figure 2 shows a configuration of a multiset $P = \{a, b, x, r_1, \ldots, r_8, s_1, \ldots, s_7\}$ of 26 points, where s_5 is repeated nine times. The closed disk $D[a, b]$ is centered at o and has diameter one, i.e., $|ab| = 1$. $D[a, b]$ contains all 8 points of the set $R = \{r_1, \ldots, r_8\}$; these points lie on the circle with radius $\frac{1}{2} - \epsilon$ that is centered at o; all points of R are in the interior of $D[a, b]$. Let $S = \{s_1, \ldots, s_7\}$ be the multiset of 15 points, where s_5 is repeated nine times. The red circles have radius 1 and are centered at points in S. Each point in S is connected to its first and second closest point (the black edges in Figure 2). Let B the chain formed by these edges. Note that r_1 and r_8 are the endpoints of B. Specifically, $|r_1s_1| = |r_8s_7| = 1$, and for each point r_i, where $2 \leq i \leq 7$, $|s_ia| > 1$, $|s_ib| > 1$, $|s_ix| > 1$, and $|r_is_{i-1}| = |r_is_i| = 1$ (here by s_5 we mean the first and last endpoints of the chain defined by points labeled s_5). Consider the Hamiltonian cycle $H = B \cup \{(r_1, a), (a, b), (b, x), (x, r_8)\}$. The longest edge in H has length 1. Therefore, the length of the longest edge in any bottleneck Hamiltonian cycle for P is at most 1. In the rest we will show—by contradiction—that any bottleneck Hamiltonian cycle of P contains (a, b). Since in B each point of S is connected to its first and second closest point, every bottleneck Hamiltonian cycle of P contains B, because otherwise, one of the points in S should be connected to a point that is farther than its second closest point, and hence that edge is longer than 1. Now we consider possible ways to construct a bottleneck Hamiltonian cycle, say H^*, using the edges in B and the points a, b, x. Assume $(a, b) \notin H^*$. Then, in H^*, a is connected to two points in $\{r_1, r_8, x\}$. We differentiate between two cases:

- $(a, x) \in H^*$. In this case $|ax| > 1$, and hence the longest edge in H^* is longer than 1, which is a contradiction.
- $(a, x) \notin H^*$. In this case $(a, r_1) \in H^*$ and $(a, r_8) \in H^*$. This means that H^* does not contain x and b, which is a contradiction.

Therefore, we conclude that H^*, and consequently any bottleneck Hamiltonian cycle of P, contains (a, b). Since $D[a, b]$ contains 8 points of $P \setminus \{a, b\}$, $(a, b) \notin$ 7-GG. Therefore 7-GG does not contain any Euclidean bottleneck Hamiltonian cycle of P.

4 Conclusion

We considered the inclusion of a Euclidean bottleneck matching and a Euclidean bottleneck Hamiltonian cycle of a point set P in higher order Gabriel graphs. It

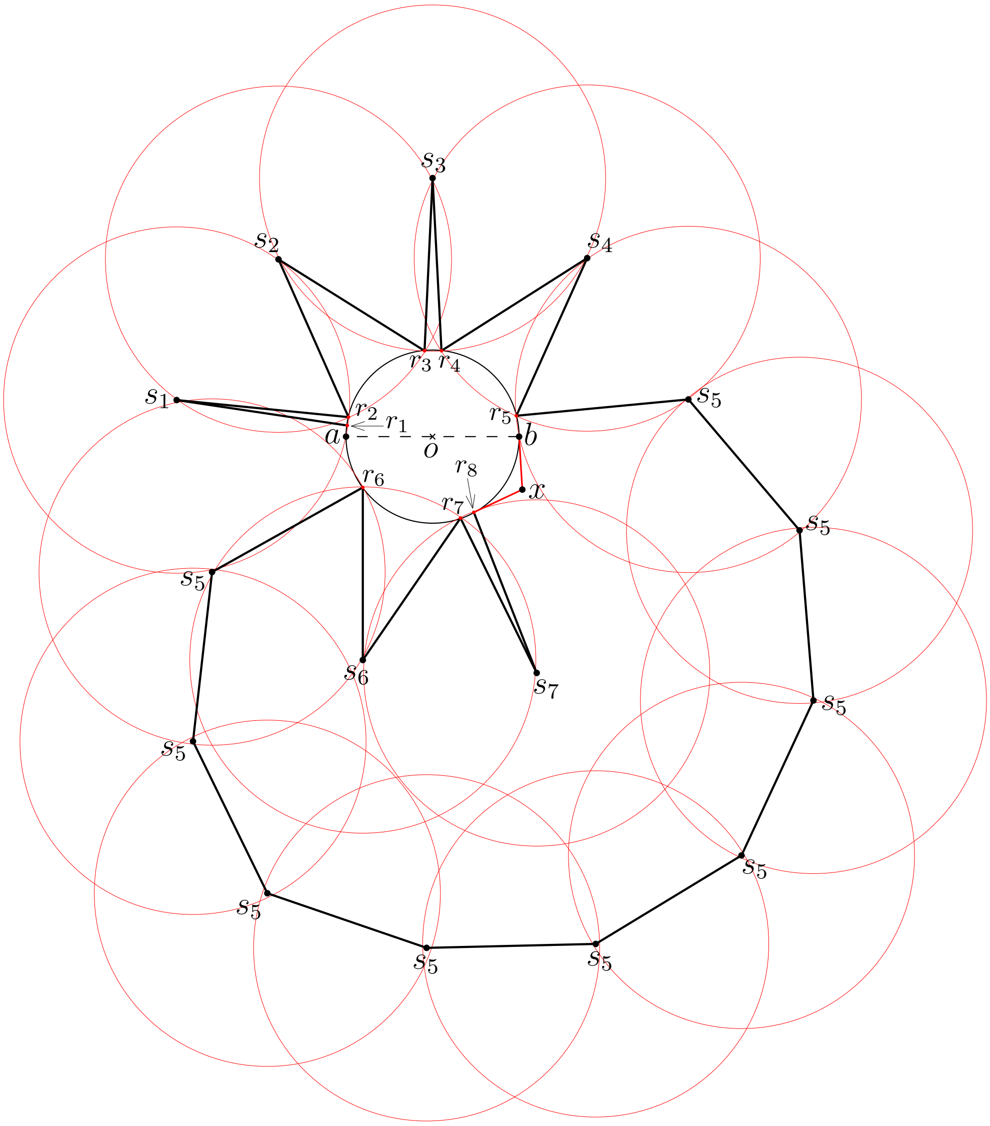

Figure 2: Proof of Proposition 1. The bold-black edges belong to B. $D[a, b]$ contains 8 points.

is known that 10-GG contains a bottleneck matching and a bottleneck Hamiltonian cycle of P. We proved that 9-GG contains a bottleneck matching of P and 7-GG may not contain any bottleneck Hamiltonian cycle of P. It remains open to decide if 8-GG or 9-GG contains any bottleneck Hamiltonian cycle of P.

References

[1] M. Abellanas, P. Bose, J. García-López, F. Hurtado, C. M. Nicolás, and P. Ramos. On structural and graph theoretic properties of higher order Delaunay graphs. *Int. J. Comput. Geometry Appl.*, 19(6):595–615, 2009.

[2] A. Biniaz, A. Maheshwari, and M. Smid. Matchings in higher-order Gabriel graphs. *Theor. Comput. Sci.*, 596:67–78, 2015.

[3] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. 20-relative neighborhood graphs are Hamiltonian. *Journal of Graph Theory*, 15(5):543–557, 1991.

[4] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. Solving the Euclidean bottleneck matching problem by k-relative neighborhood graphs. *Algorithmica*, 8(3):177–194, 1992.

[5] F. Fodor. The densest packing of 12 congruent circles in a circle. *Beiträge Algebra Geom*, 41:401–409, 2000.

[6] K. R. Gabriel and R. R. Sokal. A new statistical approach to geographic variation analysis. *Systematic Zoology*, 18(3):259–278, 1969.

[7] T. Kaiser, M. Saumell, and N. V. Cleemput. 10-Gabriel graphs are Hamiltonian. *Inf. Process. Lett.*, 115(11):877–881, 2015.

[8] D. W. Matula and R. R. Sokal. Properties of Gabriel graphs relevant to geographic variation research and the clustering of points in the plane. *Geographical Analysis*, 12(3):205–222, 1980.

[9] T.-H. Su and R.-C. Chang. The k-Gabriel graphs and their applications. In *SIGAL International Symposium on Algorithms*, pages 66–75, 1990.