From Crossing-Free Graphs on Wheel Sets to Polytopes with Few Vertices

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Abstract

A perfect matching on a finite planar point set S is crossing-free if all of its edges are disjoint in the straight-line embedding on S. In 1948 Motzkin was interested in the number of such crossing-free perfect matchings for S the $2m$ vertices of a convex polygon and he proved that to be the m-th Catalan number.

S is called a wheel set if all but exactly one point in S are vertices of its convex hull. Again we start by asking for the number of crossing-free perfect matchings of such a wheel set S, going the smallest possible step beyond Motzkins endeavor. Since position matters now, in the sense that the number is not determined by the cardinality of the wheel set alone, this immediately raises extremal and algorithmic questions. Answering these comes with all kinds of surprises. In fact, it turns out that for the purpose of counting crossing-free geometric graphs (of any type, e.g. triangulations or crossing-free spanning trees) on such a set P it suffices to know the so-called frequency vector of P (as opposed to the full order type information) – a simple formula dependent on this frequency vector exists. Interestingly, the number of order types of n points in almost convex position is roughly 2^n, compared to the number of frequency vectors which is about $2^{n/2}$.

Finally, this takes us on a journey to the rectilinear crossing-number of the complete graph, to counting of origin-embracing triangles and simplices (simplicial depth) and to counting facets of high-dimensional polytopes with few vertices.

(Based on recent joint work with Andres J. Ruiz-Vargas, Alexander Pilz, and Manuel Wettstein.)

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